On connectedness of Heegaard diagram and its cut diagram

Heegaard図とその切断図の連結性について

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§1. Introduction.

In the previous paper [7], we defined the connectedness of Heegaard diagram and its cut diagram. In that case we showed the connectedness of Heegaard diagram in relation to (Heegaard) cut diagram. The connection between them is made only by priority of cut diagram, but generally not vice versa. In this paper, we give an algorithm to construct connected cut diagram from a disconnected one. It is applied to a disconnected cut diagram of Heegaard diagram of connected sum of 3-manifolds.

We work in the piecewise linear category throughout this paper. ∂X, $\mathrm{Int}(X)$ denote the boundary, interior of a point set X, respectively. Hereafter, a closed 3-manifold M^3 denotes a connected orientable closed 3-manifold unless otherwise stated.

§2. Heegaard cut diagram.

Let (H_1, H_2, F) be a genus $n(\geqq 1)$ Heegaard splitting of M^3 so that $M^3 = H_1 \cup H_2$ and $H_1 \cap H_2 = \partial H_1 \cap \partial H_2$. Each H_1, H_2 is genus n (Heegaard) handlebody and each ∂H_1, ∂H_2 is an orientable closed common surface F (Heegaard surface) of genus n. Let $\{D_1, \cdots, D_n\}$ (resp. $\{D_1', \cdots, D_n'\}$) be a complete system of meridian disks of H_1 (resp. H_2) so that $\{D_1, \cdots, D_n\}$ (resp. $\{D_1', \cdots, D_n'\}$) is a set of pairwise disjoint properly embedded 2-disks in H_1 (resp. H_2) and the closure of $H_1-\{D_1\cup\cdots\cup D_n\}$ (resp. $H_2-\{D_1'\cup\cdots\cup D_n'\}$) is a 3-ball. Let $(H_1;\ m,\ l)$ (resp. $(H_2;\ l,\ m)$) be a genus n Heegaard diagram of (H_1, H_2, F) where $\mathrm{m}=\{m_1, \cdots, m_n\}=\{\partial D_1, \cdots, \partial D_n\}$ and $l=\{l_1, \cdots, l_n\}=\{\partial D_1', \cdots, \partial D_n'\}$. Suppose each circle l_i, m_i in ∂H_1 is oriented. If we cut off M^3 at F, then we get disjointed genus n handlebodies H_1, H_2. We put th same label F on ∂H_1 and ∂H_2. The pairwise disjointed circles $\{l_1, \cdots, l_n\}$ in ∂H_1 decompose each m_i into edges. Let $\{i_1, i_2, i_3, \cdots, i_{k_i}\}$ be the cross points of $m_i\cap(l_1\cup\cdots\cup l_n)$ $(\subset\partial H_1)$. We put labels $i_1(m_i)i_2$, $i_2(m_i)i_3$, $\cdots$, $i_{k_i}(m_i)i_1$, in these orders, to these edges in m_i according to the orientation of m_i such as $m_i=i_1(m_i)i_2\cup i_2(m_i)i_3\cup\cdots\cup i_{k_i}(m_i)i_1$. We may assume each label $i_j(m_i)i_{j+1}$ is oriented with the same orientation as m_i. The inverse orientation of $i_j(m_i)i_{j+1}$ is denoted by $i_{j+1}(m_i^{-1})i_j$. Conversely, $\{m_1, \cdots, m_n\}$ in ∂H_1 decompose each l_j into edges such as $l_j=j_1(l_j)j_2\cup j_2(l_j)j_3\cup\cdots\cup j_{l_j}(l_j)j_1$ and each label $j_i(l_j)j_{i+1}$ has the same orientation as l_j. We cut off H_1 at each D_i, then we get a 3-ball B_1^3; ∂B_1^3 is a 2-sphere S_1^2. In S_1^2, there are n pairs of 2-disks $\{D_i^+, D_i^-\}$ by cutting off H_1 at D_i. Since, both ∂D_i^+ and ∂D_i^- are decomposed by the same edges in m_i, they have oriented labels $i_1(m_i)i_2$, $i_2(m_i)i_3$, $\cdots$, $i_{k_i}(m_i)i_1$ in common. Hence we have a planar 3-regular graph which is described as a diagram over a plane $(=S_1^2-\infty)$ if a point $\infty\in S_1^2$ is designated.

Definition 1. A planar 3-regular graph

$$\begin{aligned}\{\partial D_i^+ &= i_1(m_i)i_2\cup i_2(m_i)i_3\cup\cdots\cup i_{k_i}(m_i)i_1,\\ \partial D_i^- &= i_1(m_i)i_2\cup i_2(m_i)i_3\cup\cdots\cup i_{k_i}(m_i)i_1,\\ &\{j_1(l_j)j_2,\ j_2(l_j)j_3,\ \cdots,\ j_{l_j}(l_j)j_1\}\}\ (i,\ j=1,\cdots,n)\end{aligned}$$

is called the *Heegaard cut diagram* (*cut diagram*) associated with $(H_1;\ m,$

l) and is described as $G(m, l)$. Similarly, $G(l, m)$ associated with $(H_2; l, m)$ is defined and its expression is

$$G(l, m)=\{\partial D_j'^{+}=j_1(l_j)j_2\cup j_2(l_j)j_3\cup\cdots\cup j_{l_j}(l_j)j_1,$$
$$\partial D_j'^{-}=j_1(l_j)j_2\cup j_2(l_j)j_3\cup\cdots\cup j_{l_j}(l_j)j_1,$$
$$\{i_1(m_i)i_2, i_2(m_i)i_3, \cdots, i_{k_i}(m_i)i_1\}\}\ (i, j=1,\cdots,n).$$

A pair $G(m, l)\cup G(l, m)$ is called the *pair* of cut diagrams of $(H_1; m, l)$ and $(H_2; l, m)$.

Proposition 1. *There exist the same labelled four vertices, the same oriented labelled three edges and the same oriented labelled two faces, i.e. 2-disks or punctured 2-disks*[1] *in* $G(m, l)\cup G(l, m)$.

Proof. Let $|G(m, l)|$ be the presentation of the underlying space of labels of $G(m, l)$. Let $\sigma_1, \cdots, \sigma_p$ be faces that is the closures of connected components of $S_1^2-|G(m, l)|-\bigcup_{j=1}^{n}(D_j^{+}\cup D_j^{-})$. And let the name σ_i of face be the label of its. Since $\partial H_1=F$, $\partial H_2=F$, there exists the face which should be put on the same label σ_i in $G(l, m)$. See Fig. 1. $m\cap l$[2] consist of cross points of $(H_1; m, l)$ or $(H_2; l, m)$. Since there are two edges $i_k(m_i)i_{k+1}(\subset m_i=\partial D_i^{+}, \partial D_i^{-})$ in $G(m, l)$, there exist the same labelled two points i_k in $G(m, l)$. Since there is $i_k(m_i)i_{k+1}$ in $G(l, m)$, there exist $D_j'^{+}$, $D_j'^{-}$ so that $i_k\subset\partial D_j'^{+}$ and $i_k\subset\partial D_j'^{-}$ in $G(l, m)$. We put the same label D_i (resp. D_j') on the two disks D_i^{+}, D_i^{-} (resp. $D_j'^{+}$, $D_j'^{-}$) in $G(m, l)$ (resp. $G(l, m)$) $(i, j=1,\cdots,n)$. Hence by counting the same labels of the vertices, edges and faces in $G(m, l)\cup G(l, m)$, proposition is proved. □

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1 disk with $n(\geqq 1)$ holes.

2 $m\cap l=(m_1\cup m_2\cup\cdots\cup m_n)\cap(l_1\cup l_2\cup\cdots\cup l_n)$

Let $(H_1; m, l)=(H_1; m_1, \cdots, m_n, l_1, \cdots, l_n)$ be a genus $n(\geqq 2)$ Heegaard diagram of (H_1, H_2, F) and $G(m, l)$ the same presentation as in definition 1. We choose $n-1$ pieces of edges from $\{j_i(l_j)j_{i+1}\}$ in $\{l_j\}$ $(j=1,\cdots,n)$ of $G(m, l)$ and let these edges be $\{L_1, \cdots, L_{n-1}\}$.

Definition 2. $(H_1; m, l)$ is called *connected* for the meridian system m if $\{L_1, \cdots, L_{n-1}\}$ can be chosen so that $(L_1 \cup \cdots \cup L_{n-1}))\cup(m_1 \cup \cdots \cup m_n)$ becomes a connected graph. If $(H_1; m, l)$ is not connected, then it is called *disconnected* for m. The connected orientable closed 3-manifolds which have genus 1 Heegaard diagram are 3-sphere S^3, lens spaces $L(p, q)$ and $S^2\times S^1$. We define that each genus 1 Heegaard diagram of S^3 and $L(p, q)$ is connected and genus 1 Heegaard diagram of $S^2\times S^1$ is disconnected for m $(=\{m_1\})$.

Definition 3. Let $G(m, l)$ be the same presentation as in definition 1. If the closures of connected components of $S_1^2-|G(m, l)|$ consist of 2-disks, then $G(m, l)$ is called *connected* for the meridian system m. If $G(m, l)$ is not connected, then $G(m, l)$ is called *disconnected* for m.

Example 1. In the figure 1, there are genus 2 Heegaard diagrams $(H_1; m, l)$, $(H_2; l, m)$ of (H_1, H_2, F) of the 3-sphere S^3. l_1 and m_2 are drawn heavy. Both $(H_1; m, l)$ and $(H_2; l, m)$ are connected. $G(m, l)$ and $G(l, m)$ are cut diagrams of $(H_1; m, l)$ and $(H_2; l, m)$, respectively. Both $G(m, l)$ and $G(l, m)$ are connected.

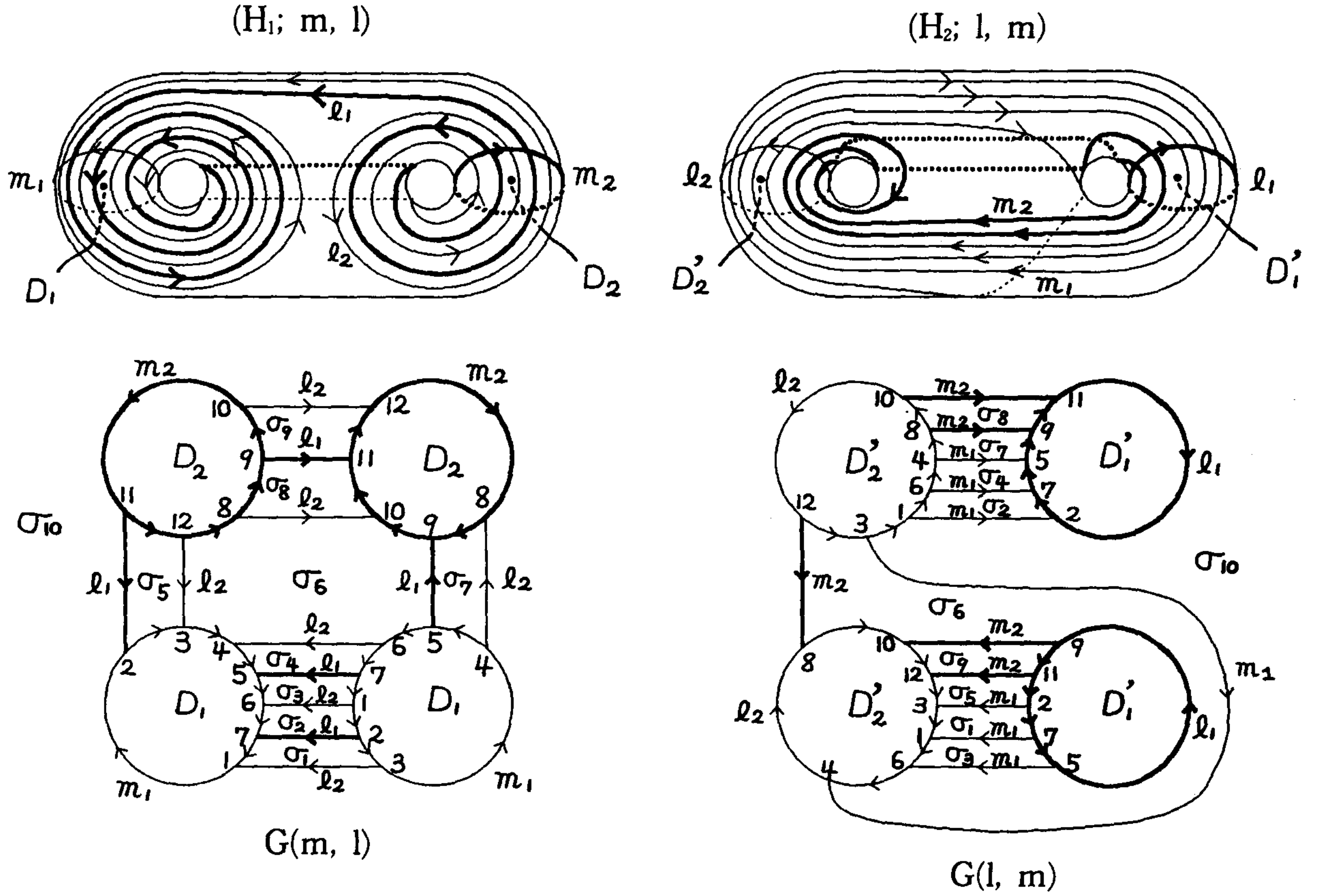


Fig. 1

Example 2. Figure 2 shows a disconnected $G(m, l)$ of $(H_1; m, l)$ of $S^2 \times S^1$ But here the connection of Heegaard diagram $(H_1; m, l)$ is made.

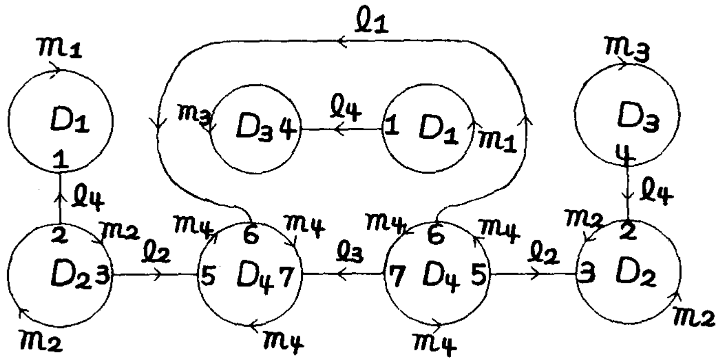


Fig. 2

From the above example, we have

Proposition 2. If $G(m, l)$ of $(H_1; m, l)$ is connected, then $G(l, m)$ of $(H_2; l, m)$ also becomes connected. If $G(m, l)$ of $(H_1; m, l)$ is connected, then $(H_1; m, l)$ also becomes connected. But the reverse of this does not hold generally. If Heegaard genus equals to 1, then the connectedness of $(H_1; m_1, l_1)$ is equivalent to that of $G(m_1, l_1)$.

§3. Construction of connected $G(m, l)$ from disconnected one.

We need the next deformations to transform a cut diagram.

Definition 4.

(1) (G-1) $\Rightarrow$ (G-2) is called *D_3^+-deformation* and (G-2) $\Rightarrow$ (G-1) is called *D_3^--deformation*. They are generally called *D_3-deformations*.

(2) (G-3) $\Rightarrow$ (G-4) is called *D_2^+-deformation* and (G-4) $\Rightarrow$ (G-3) is called *D_2^--deformation*. They are generally called *D_2-deformations*.

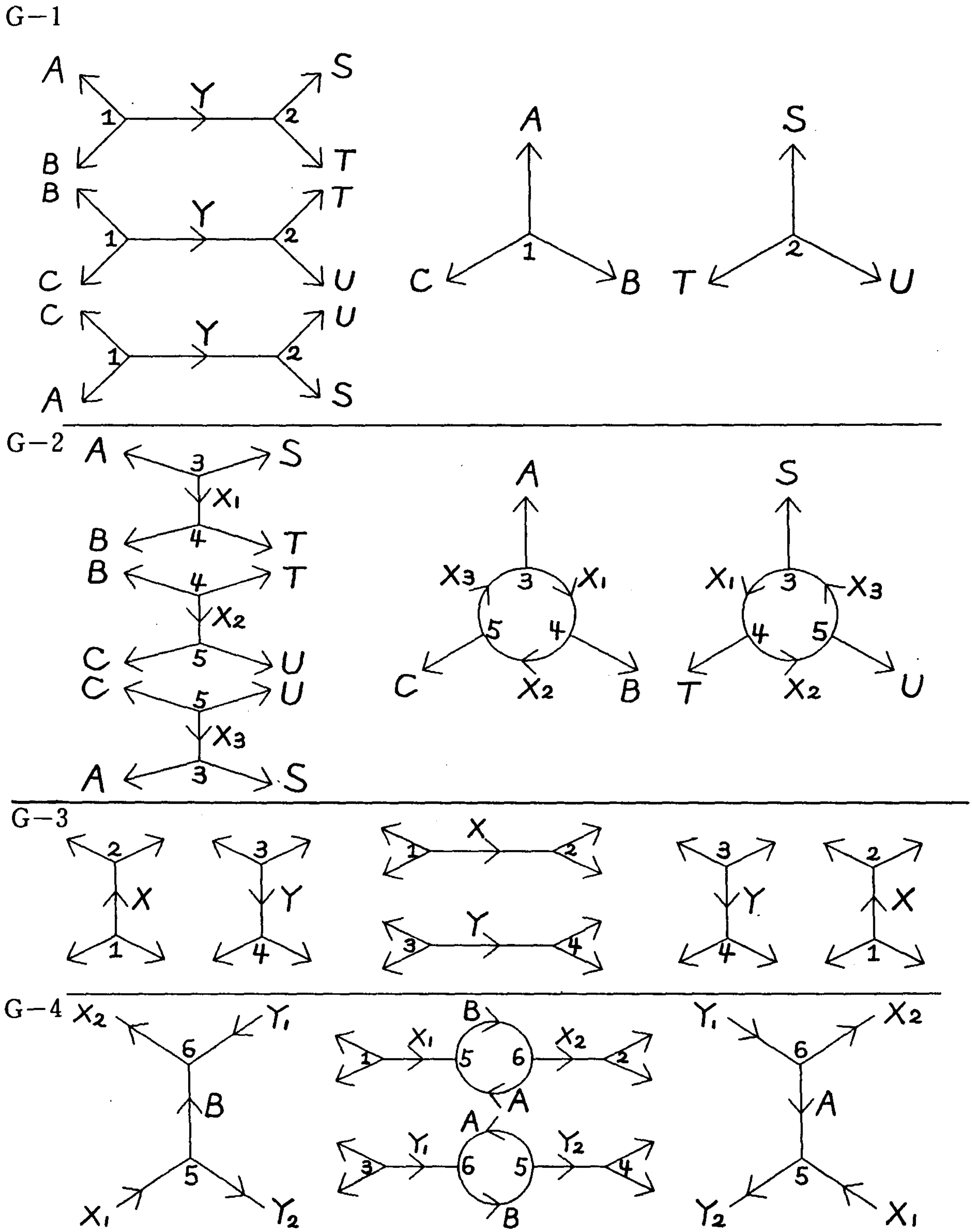


Fig. 3

(1) and (2) are generally called the *elementary DS-deformations* for DS-diagram [3].

The elementary DS-deformations preserve homeomorphism of a 3-manifold.

The proposition 1 shows that $G(m, l) \cup G(l, m)$ has the same structure of a DS-diagram without any connectedness (components of a DS-diagram are one and those of $G(m, l) \cup G(l, m)$, more than one). Hence we can directly apply the elementary DS-deformations to Heegaard cut diagrams $G(m, l) \cup G(l, m)$.

Lemma 1. *Let $(H_1; m, l)$, $(H_2; l, m)$ be genus $n(\geqq 1)$ Heegaard diagrams of (H_1, H_2, F) of M^3 and $G(m, l)$, $G(l, m)$, the cut diagrams, respectively. Let $G(m, l)$, $G(l, m)$ be the same presentations as in definition 1, respectively. Applying the elementary DS-deformations to $G(m, l) \cup G(l, m)$, one can construct genus $n+1$ or $n+2$ cut diagrams.*

Proof. Let ξ be a face in the closures of connected components of $S_1^2 - |G(m, l)| - \bigcup_{j=1}^{n} (D_j \cup D_j)$. Since $F = \partial H_1$, $F = \partial H_2$, there exists the same labelled face ξ in $G(l, m)$ which is inversely oriented to that in $G(m, l)$. We draw same oriented labelled edges X in both ξ which satisfy the following conditions. Let $\{P_1, P_2\}$ be two points of ∂X.

(C1) In the case that we can take $\{l_{\alpha\beta}, l_{\gamma\delta}\}$ $(\alpha \neq \gamma)$ so that $P_1 \subset \mathrm{Int}\,(l_{\alpha\beta}) \subset \partial\xi$ and $P_2 \subset \mathrm{Int}\,(l_{\gamma\delta}) \subset \partial\xi$. See Fig. 4.

We act D_2^+-deformation shown as the dotted lines and circles in $\{G(m, l) \cup X\} \cup \{G(l, m) \cup X\}$. Then we get deformed diagrams in Fig. 5.

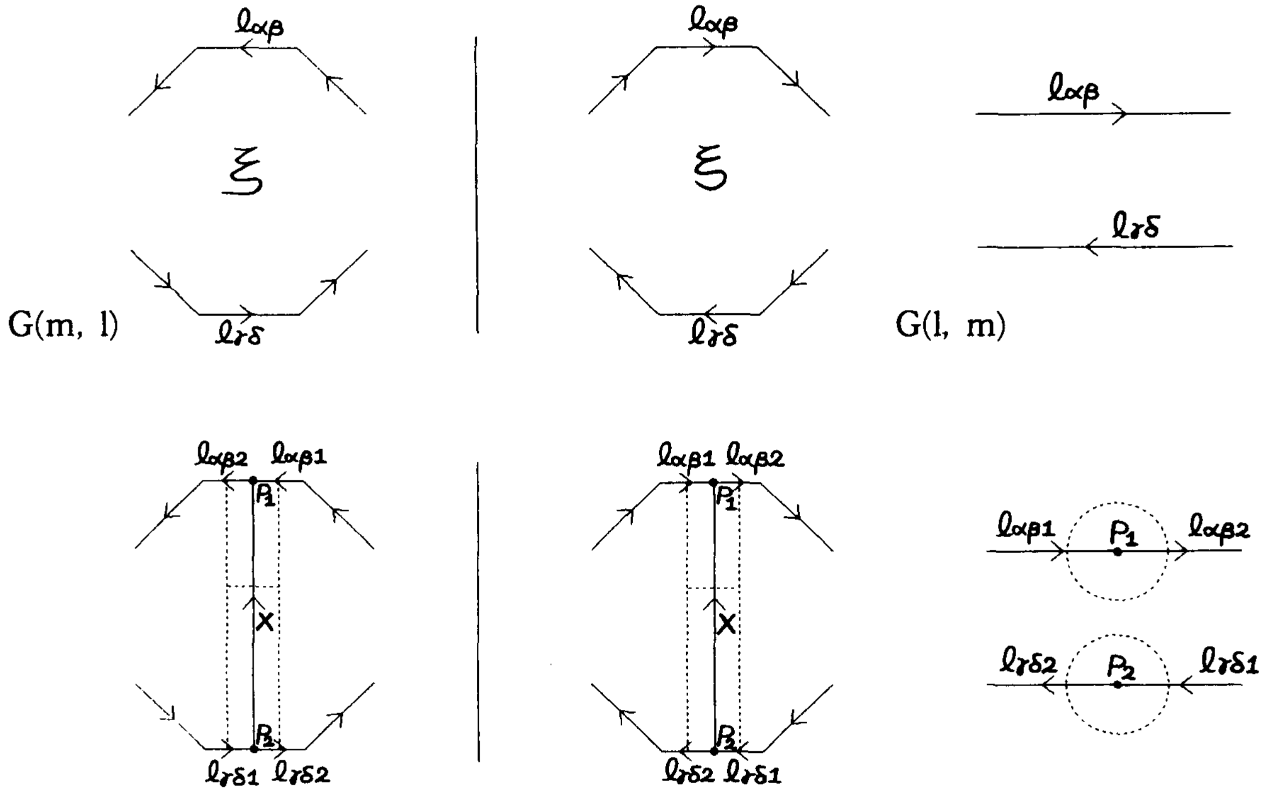


Fig. 4

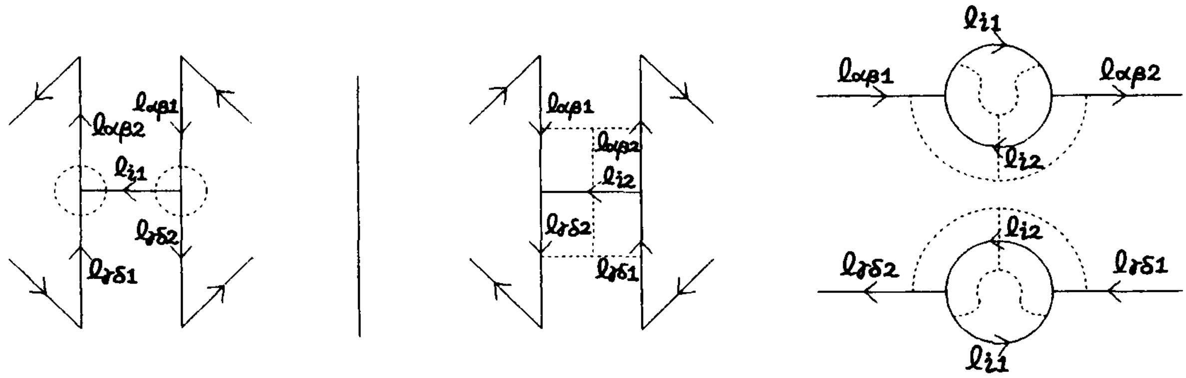
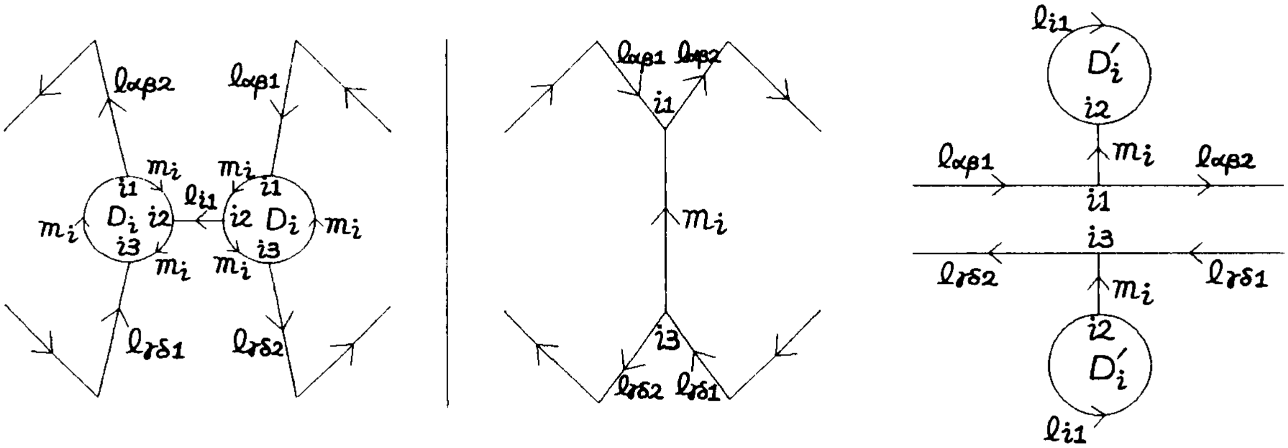


Fig. 5

Next we act D_3^+-deformation to l_{i1}, l_{i2} shown as the dotted lines and circles in Fig. 5, then we get genus $n+1$ cut diagrams $G_1(m, l) \cup G_1(l, m)$ in Fig. 6.



$G_1(m,\ l)$　　　　　　　　　　　　$G_1(l,\ m)$

Fig. 6

(C2)　In the case that we can not take $\{l_{\alpha\beta},\ l_{\gamma\delta}\}$ $(\alpha\neq\gamma)$ so that $P_1\subset$ Int $(l_{\alpha\beta})\subset\partial\xi$ and $P_2\subset$Int $(l_{\gamma\delta})\subset\partial\xi$.

Let $(U;\ m,\ l)$, $(V;\ l,\ m)$ be the genus 1 Heegaard diagrams of $(U,\ V,\ T)$ of S^3 and $G_U(m,\ l)$, $G_V(l,\ m)$ the cut diagrams, respectively. We draw $G_U(m,\ l)$ (resp. $G_V(l,\ m)$) in Int (ξ) $(\subset |G(m,\ l)|)$ (resp. Int (ξ) $(\subset |G(l,\ m)|)$). This means that we construct the connected sum $M^3\# S^3$ [3] of M^3 and S^3. We draw X in both $\xi-$Int $(|G_U(m,\ l)|)$, $\xi-$Int $(|G_V(l,\ m)|)$ which satisfy the condition (C1). And act the same operations as in (C1), then we get genus $n+2$ cut diagrams of M^3. □

Remark 1.　In (C1), we may change $\{m_{\alpha\beta},\ m_{\gamma\delta}\}$ $(\alpha\neq\gamma)$ instead of $\{l_{\alpha\beta},\ l_{\gamma\delta}\}$. But if we take a pair $\{l_{\alpha\beta},\ m_{\gamma\delta}\}$ instead of $\{l_{\alpha\beta},\ l_{\gamma\delta}\}$, then we can not construct cut diagrams from $\{G(m,\ l)\cup X\}\cup\{G(l,\ m)\cup X\}$.

Remark 2.　The diagrams shown in Fig. 5 are not Heegaard cut diagrams.

[3] $M^3\# S^3$ is homeomorphic to M^3

Using the above lemma, we have

Theorem 1. *Let $G(m, l) \cup G(l, m)$ be disconnected genus $n(\geqq 1)$ cut diagrams of (H_1, H_2, F) of M^3. Then one can construct connected cut diagrams from the disconnected ones.*

Proof. We give an algorithm of construction as follows.

Step 1. Let ξ be not 2-disks in both $G(m, l)$, $G(l, m)$. We can draw the same oriented labelled edges $X_1, \cdots, X_t$ in both ξ which satisfy the following conditions:

(1) Let each X_i be taken as in (C1) or (C2) in lemma 1.

(2) Let $\{P_{i1}, P_{i2}\}$ be two points of ∂X_i and $\{P_{i1}, P_{i2}\}$ $(i=1,\cdots,t)$ differently.

(3) The closures of connected components of $\xi-(X_1 \cup \cdots \cup X_t)$ become 2-disk.

Step 2. Act the operations under (C1) to the part of both X_1 in $\{G(m, l) \cup X_1\} \cup \{G(l, m) \cup X_1\}$. Then we get genus $n+1$ or $n+2$ cut diagrams of M^3.

Step 3. Act the same operations as step 2 to other X_i $(i=2,\cdots,t)$, then we get cut diagrams which genus are more than $n+t-1$. Let the cut diagrams be $G'(m, l) \cup G'(l, m)$.

Step 4. If $G'(m, l)$, $G'(l, m)$ are not connected, then act the same operations from step 1 to 3 to $G'(m, l) \cup G'(l, m)$, repeatedly, then we can get connected cut diagrams. □

Let $(U_i; m, l)$, $(V_i; l, m)$ $(i=1,2)$ be genus $n_i(\geqq 1)$ Heegaard diagrams of (U_i, V_i, F_i) of M_i^3 and $G_i(m, l) \cup G_i(l, m)$ the cut diagrams, respectively. If we construct the connected sum $M_1^3 \# M_2^3$ of M_1^3 and M_2^3, then we get a disconnected genus n_1+n_2 Heegaard diagrams and its disconnected cut diagrams. Then we have

Corollary 1. *From the disconnected cut diagrams* $\{G_1(m, l) \cup G_2(m, l)\} \cup \{G_1(l, m) \cup G_2(l, m)\}$ *of* $M_1^3 \# M_2^3$, *one can construct connected cut diagrams.*

§4. Examples.

We give two examples.

Example 3. The genus 1 cut diagrams $G(m, l) \cup G(l, m)$ of $S^2 \times S^1$ are given in Fig. 7. We will construct connected cut diagrams from them. Here we need many pictures.

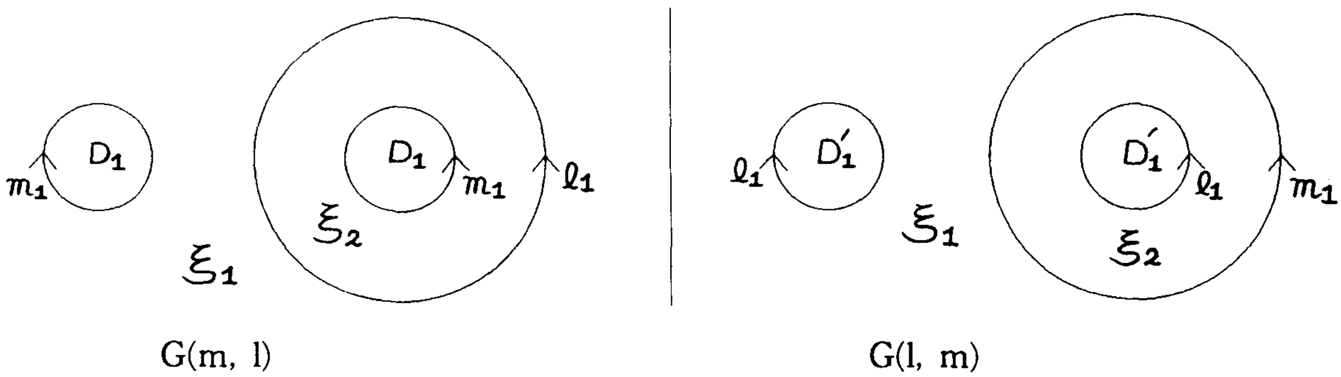
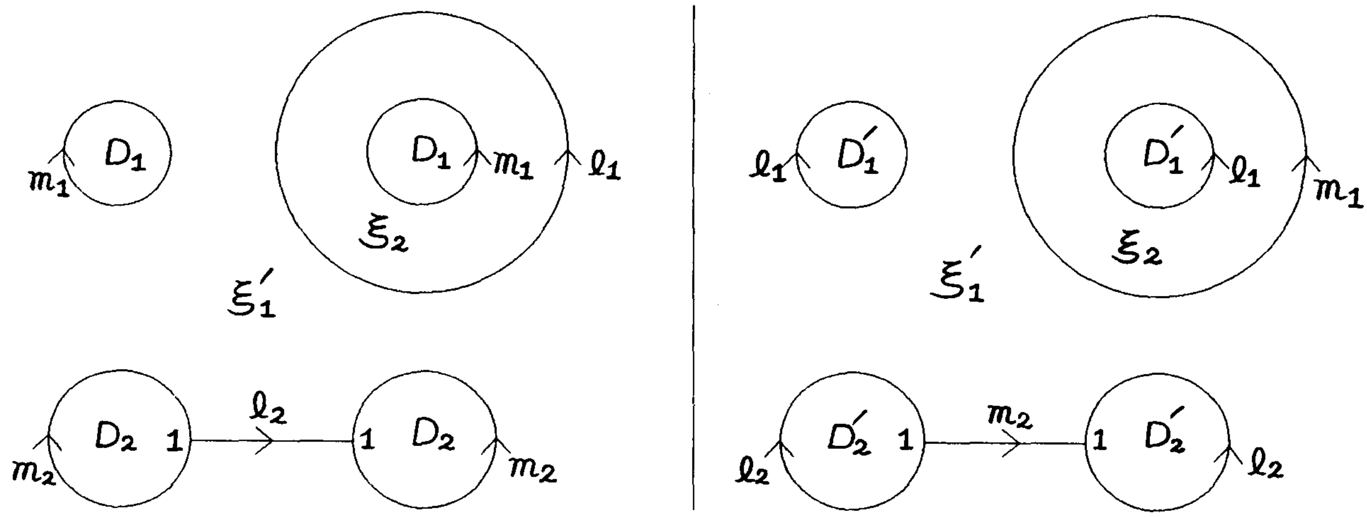


Fig. 7

Let ξ_1, ξ_2 be not 2-disks, respectively. Take the genus 1 Heegaard diagrams of S^3 in both ξ_1. Then we have genus 2 cut diagrams $G_1(m, l) \cup G_1(l, m)$ in Fig. 8. This means that we construct the connected sum $(S^2 \times S^1) \# S^3$ of $S^2 \times S^1$ and S^3. Let ξ_1' be the closure of $\xi_1 - (D_2 \cup l_2 \cup D_2)$

(resp. $\xi_1 - (D_2' \cup m_2 \cup D_2')$).



$G_1(m, l)$ $G_1(l, m)$

Fig. 8

Take two edges X_1, X_2 in both ξ_1' shown in Fig. 9. Let ξ_1'' be the closure of $\xi_1' - (D_2 \cup l_2 \cup D_2 \cup X_1 \cup X_2)$ (resp. $\xi_1' - (D_2' \cup m_2 \cup D_2' \cup X_1 \cup X_2)$). Then both ξ_1'' become 2-disks.

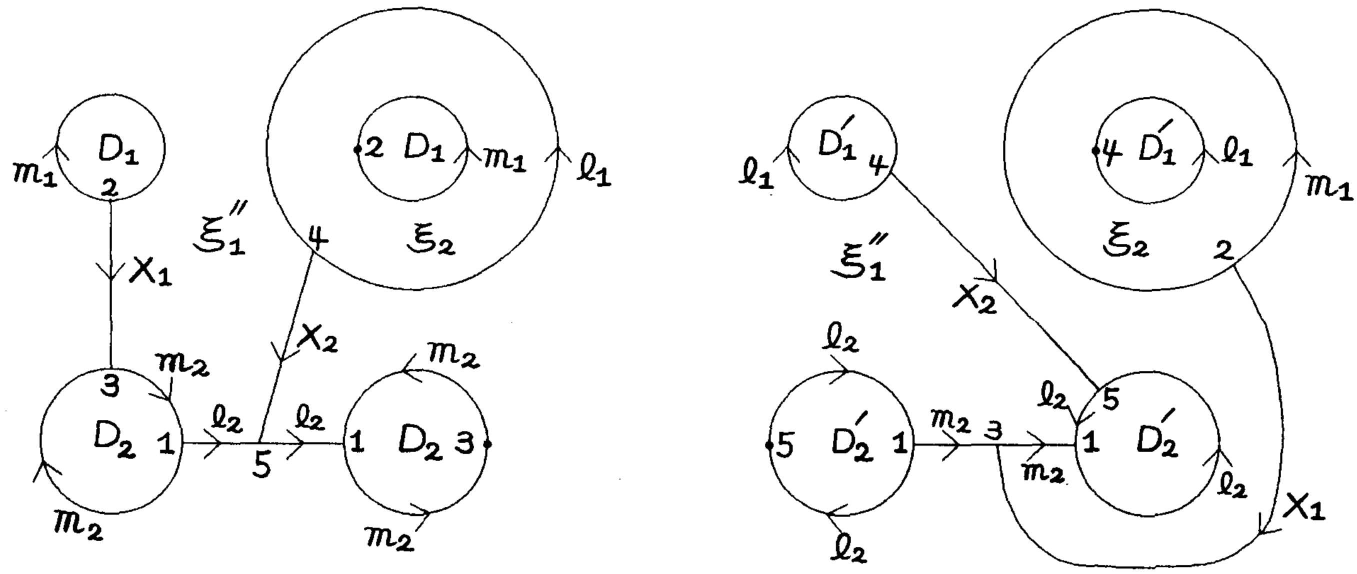


Fig. 9

We act D_2^+-deformations to each X_1, X_2 shown as the dotted lines and circles in Fig. 10. Then we get Fig. 11.

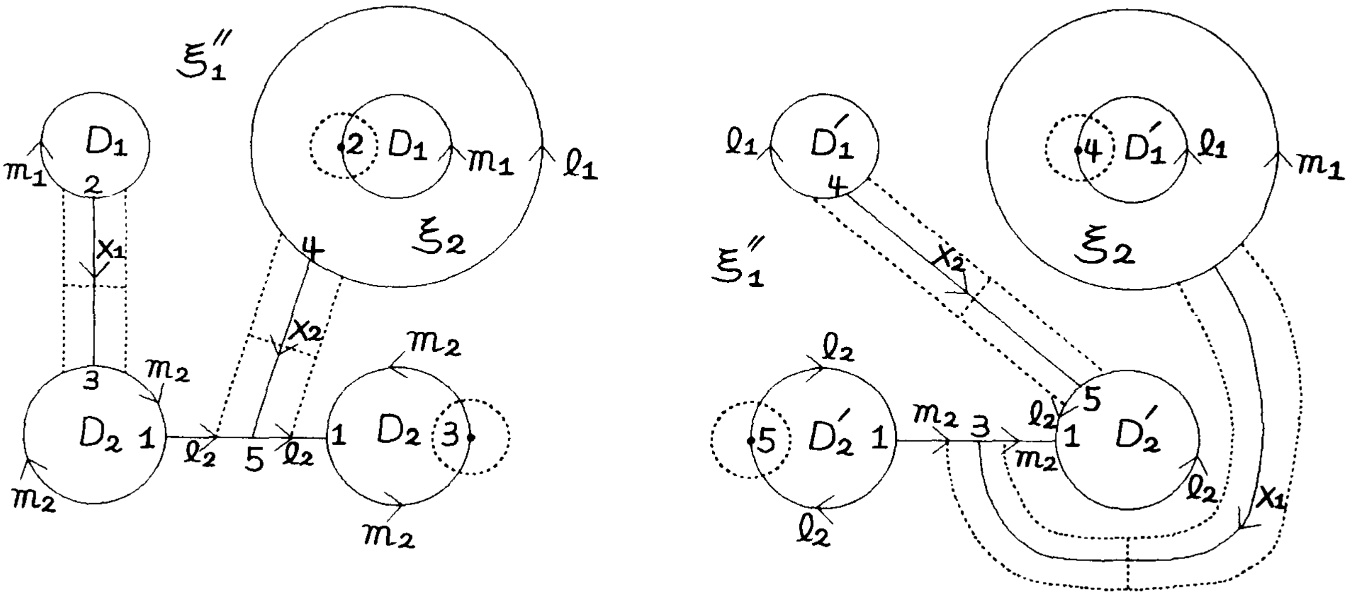


Fig. 10

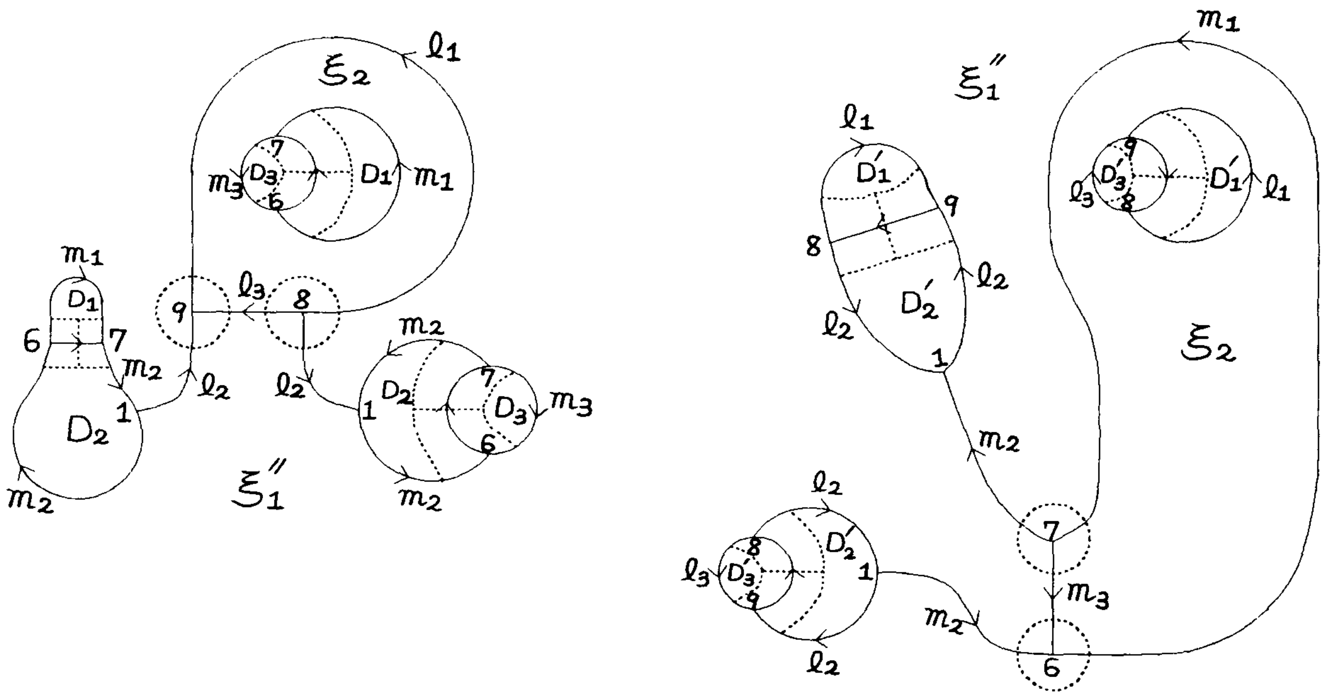


Fig. 11

Further, if we act D_3^+-deformations shown as the dotted lines and circles in Fig. 11, then we get genus 4 disconnected cut diagrams $G_2(m, l) \cup G_2(l, m)$ of $S^2 \times S^1$ in Fig. 12. $G_2(m, l)$ is the same diagram as $G(m, l)$ in Fig. 2.

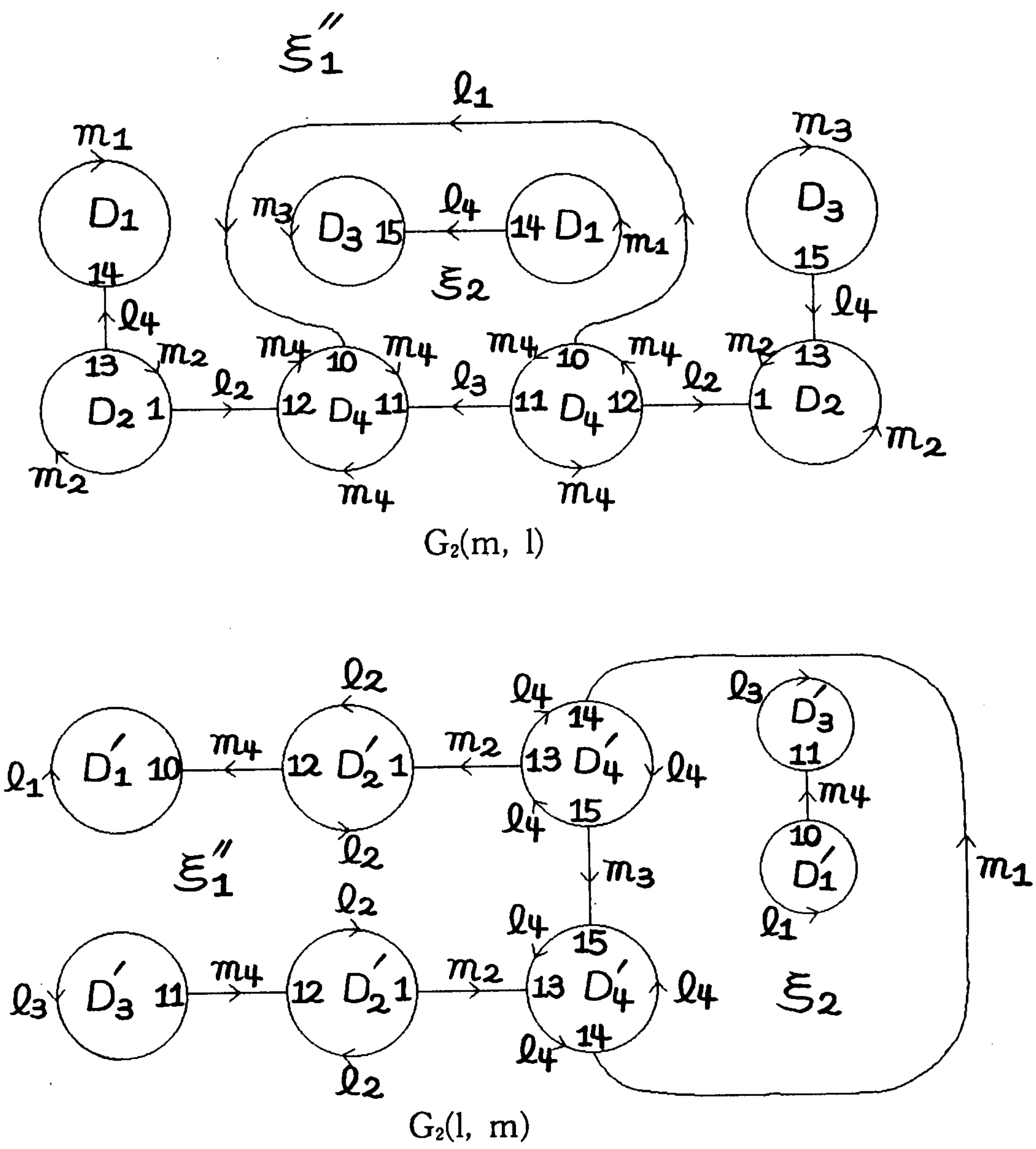


Fig. 12

Take X_3 in both ξ_2 as shown in Fig. 13. Let ξ_2' be the closure of ξ_2- $(D_3 \cup l_4 \cup D_1 \cup X_3)$ (resp. $\xi_2-(D_3' \cup m_4 \cup D_1' \cup X_3)$). Then both ξ_2' become 2-disks.

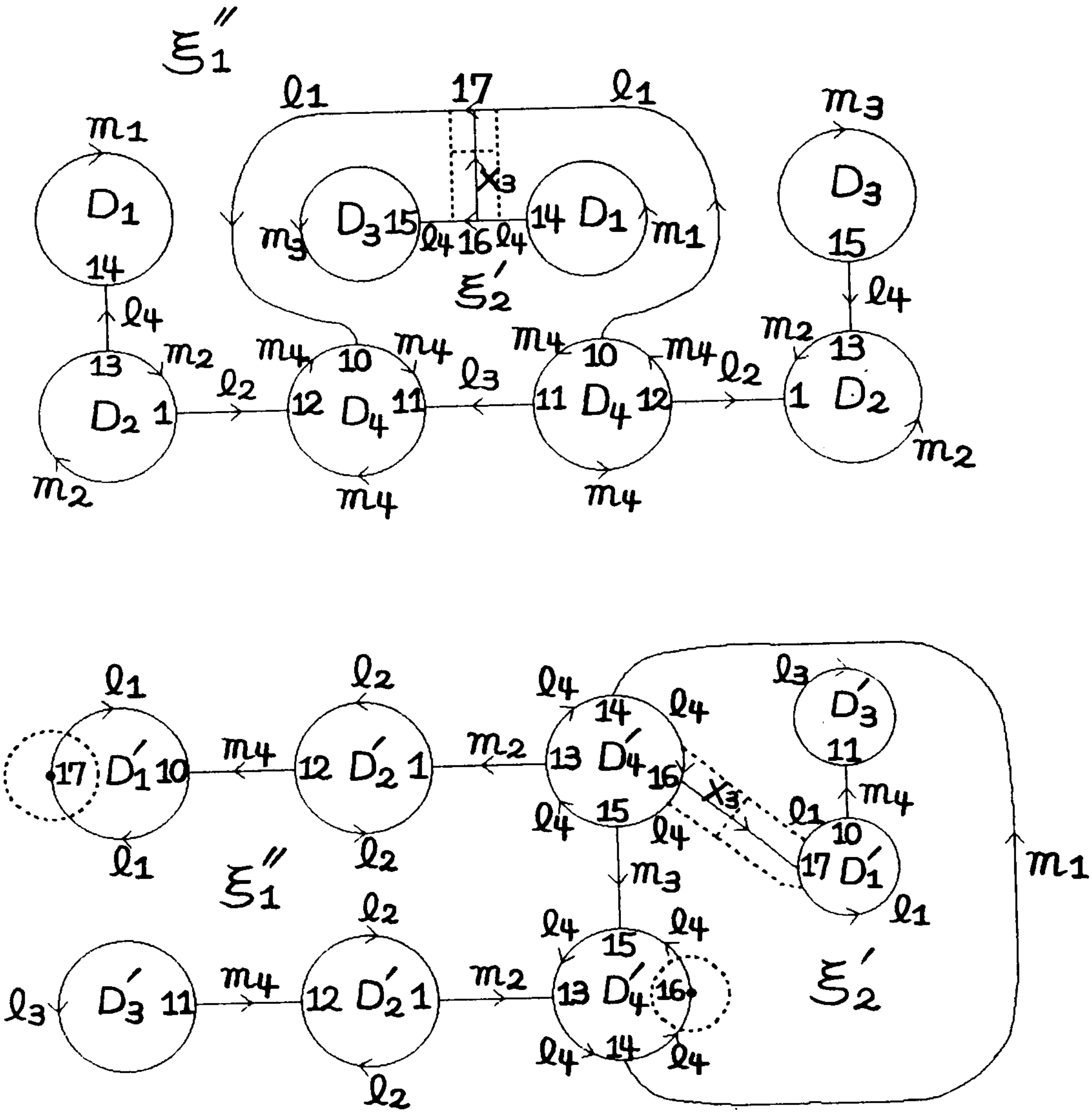


Fig. 13

Act D_2^+-deformation to both X_3 in Fig. 13. Then we get Fig. 14.

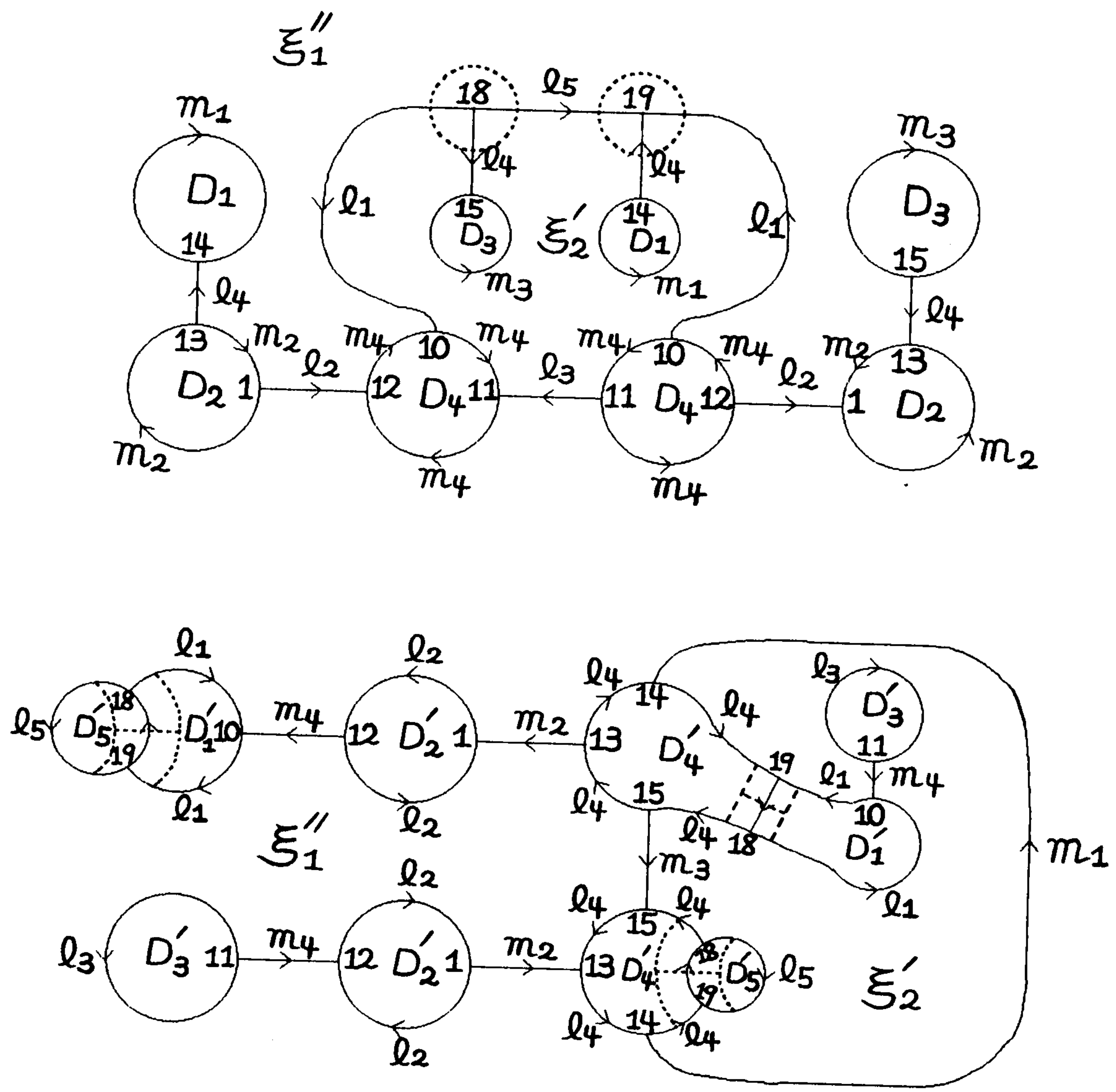
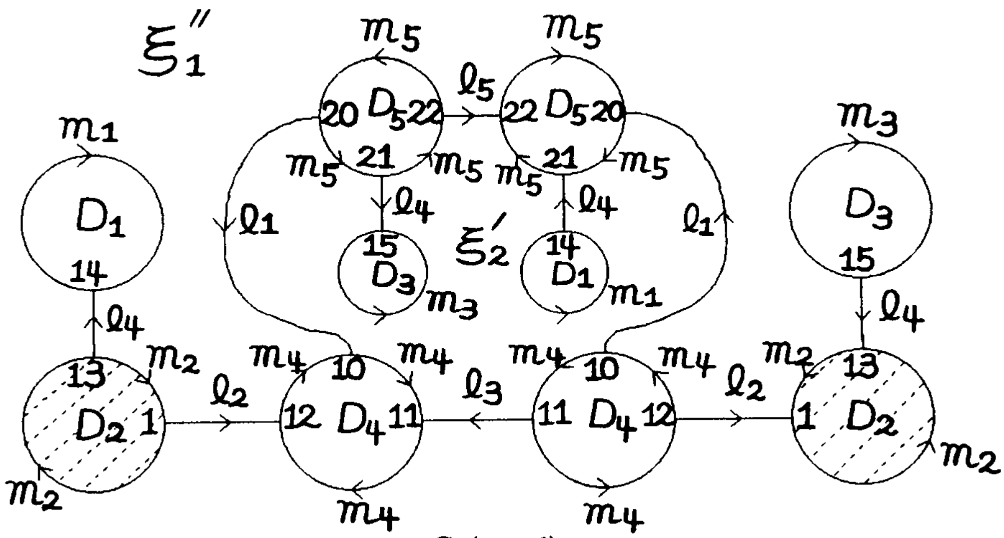
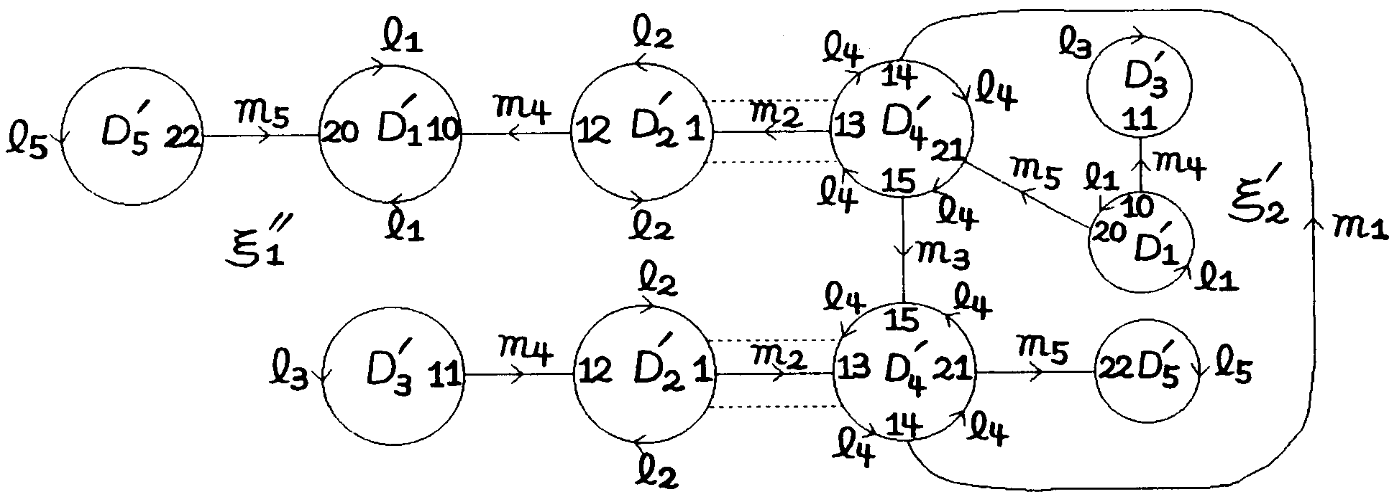


Fig. 14

Act $D_3{}^{+}$-deformation to the diagrams in Fig. 14, then we get connected genus 5 cut diagrams $G_3(m, l) \cup G_3(l, m)$ of $S^2 \times S^1$ in Fig. 15.



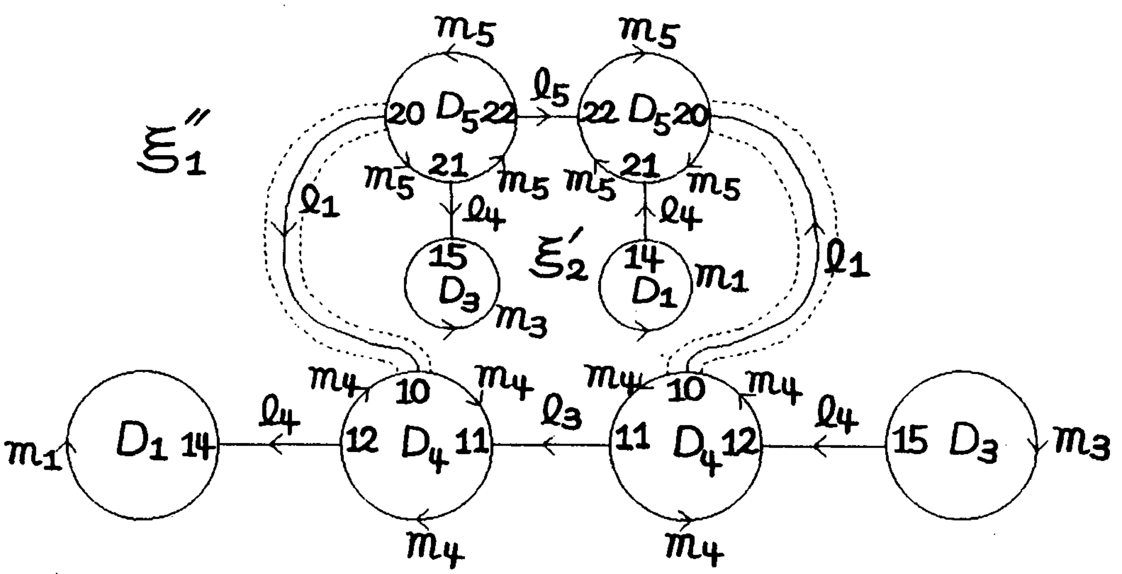
G3(m, l)



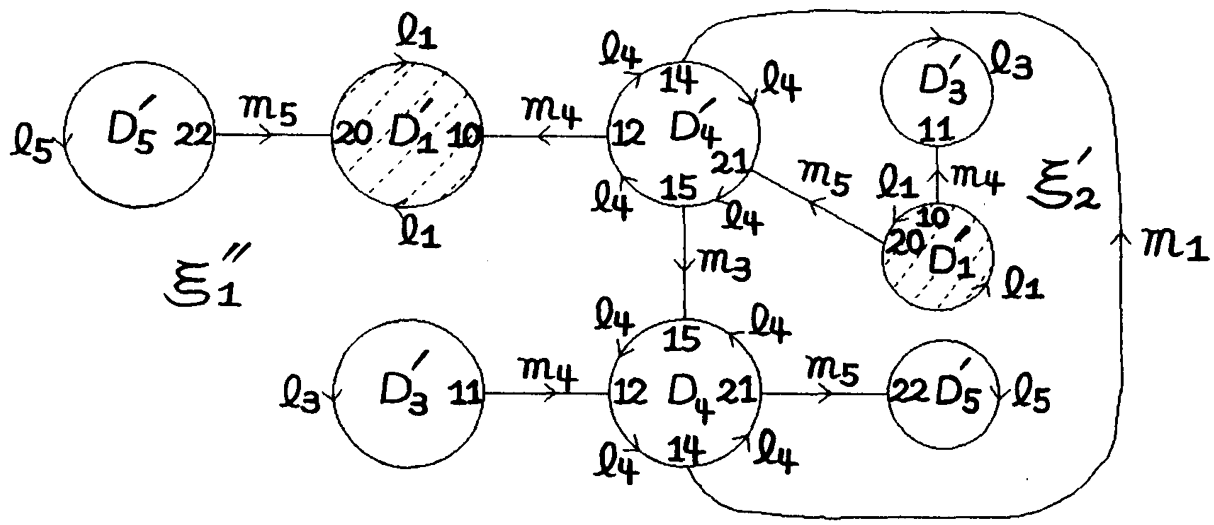
G3(l, m)

Fig. 15

Further, by D_2^--deformation, we can get genus 4 connected cut diagrams $G_4(m, l) \cup G_4(l, m)$ of $S^2 \times S^1$ in Fig. 16.



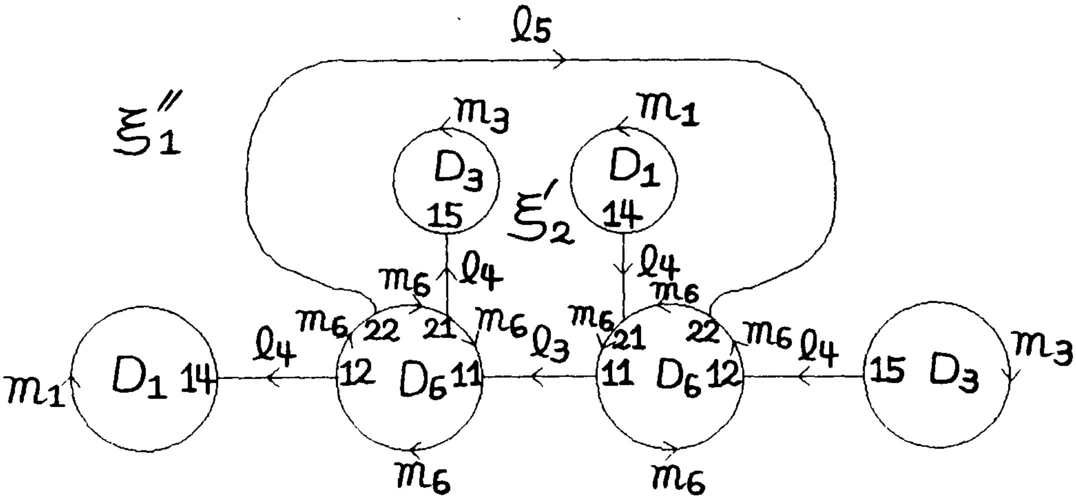
G₄(m, l)



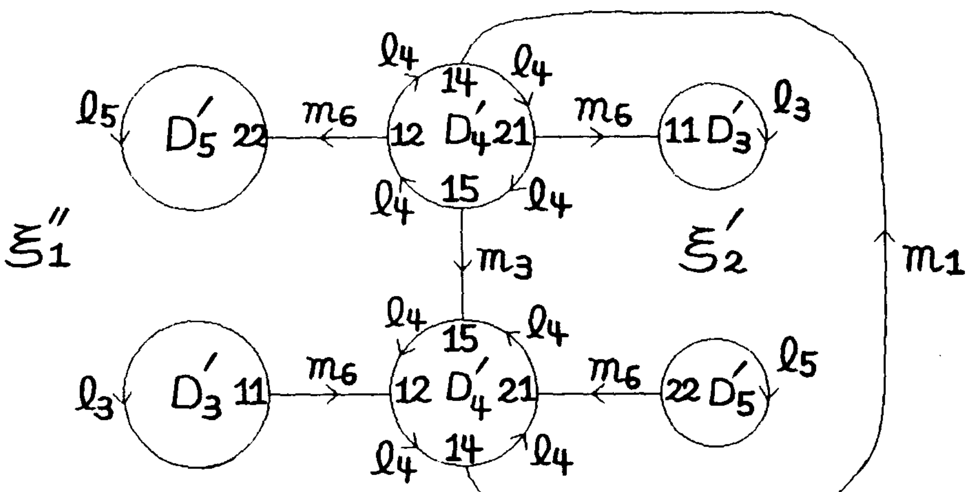
G₄(l, m)

Fig. 16

Further, by D_2^--deformation, we can get genus 3 connected cut diagrams $G_5(m, l) \cup G_5(l, m)$ of $S^2 \times S^1$ in Fig. 17.



G_5(m, l)



G_5(l, m)

Fig. 17

Example 4. Fig. 18 gives a disconnected genus 2 Heegaard diagram $(H_1;\ m,\ l)$ of $L(7,\ 2)\#L(7,\ 4)$ and its disconnected cut diagrams $G(m,\ l)$[4] and $G(l,\ m)$ of $(H_2;\ l,\ m)$. We omit the picture $(H_2;\ l,\ m)$ in the figure.

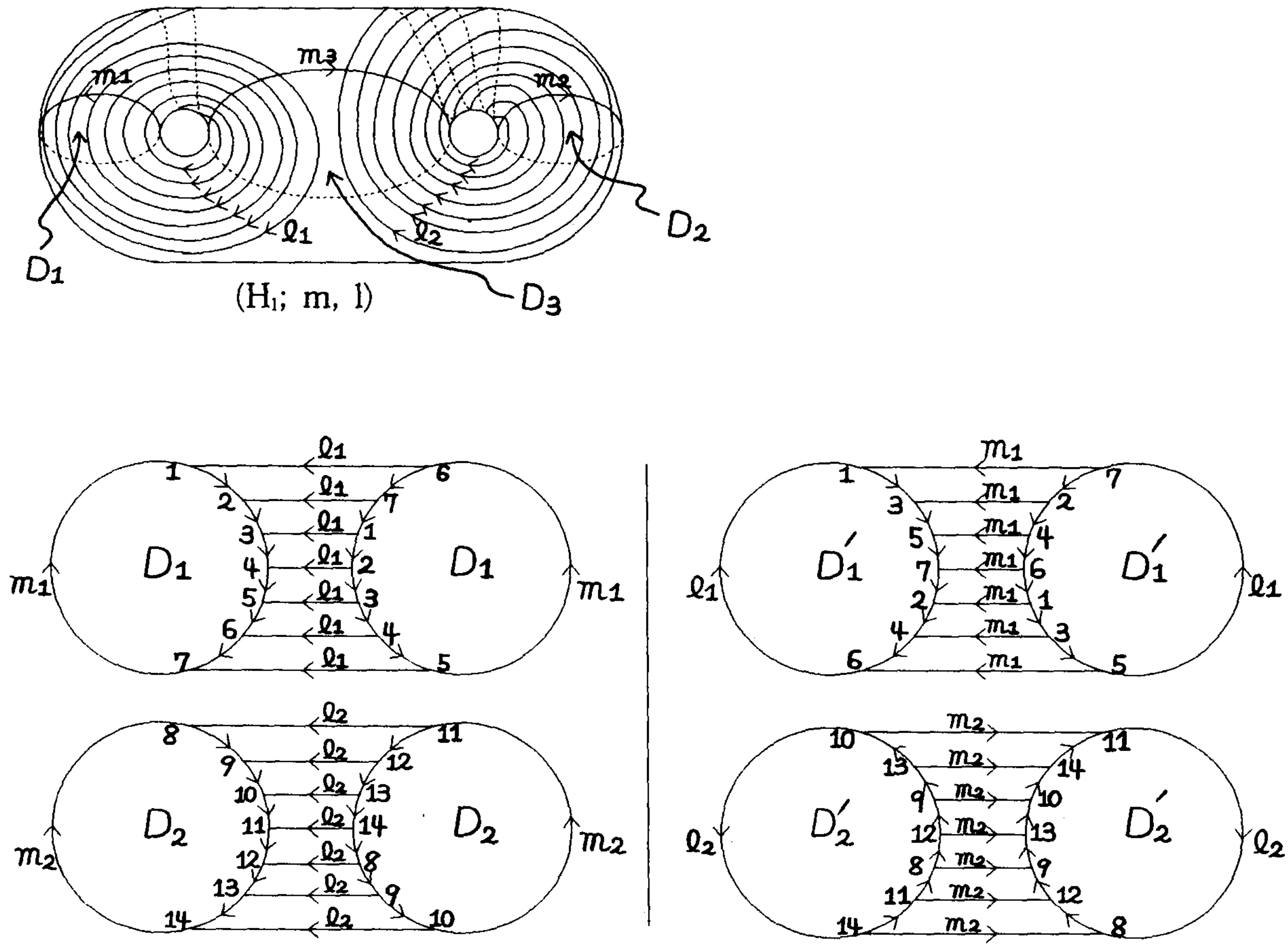
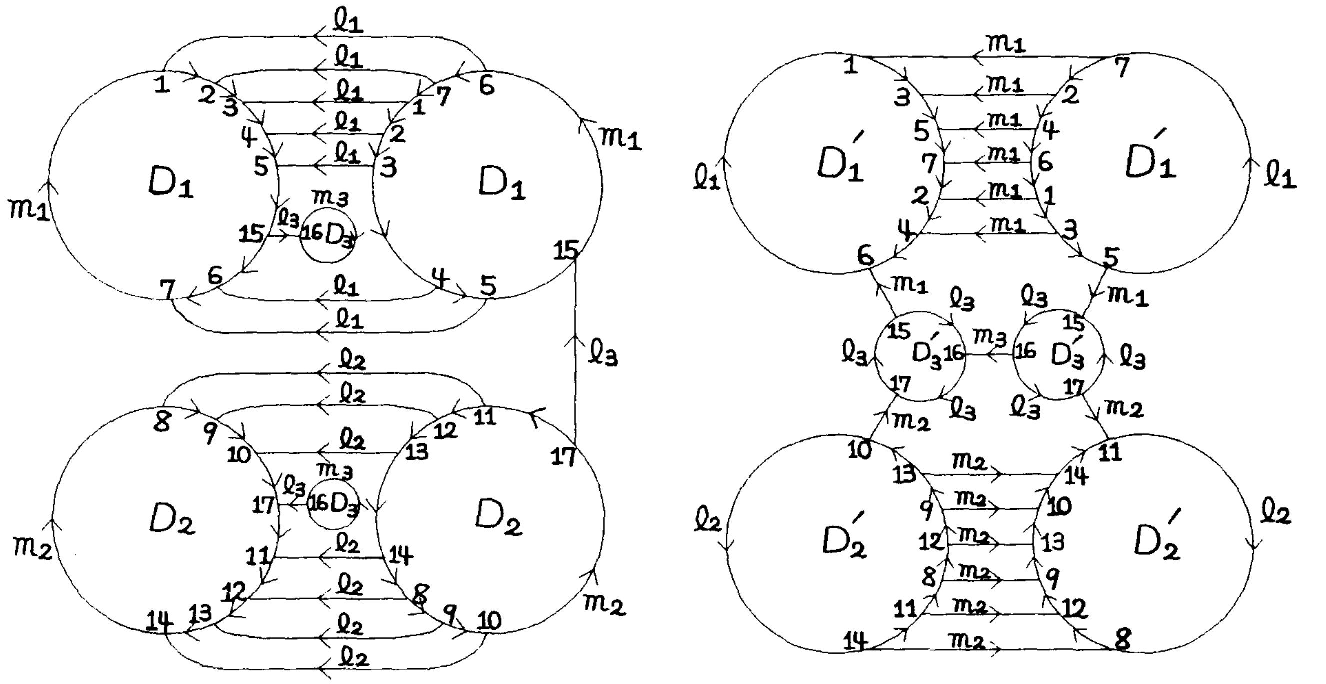


Fig. 18

4 Note that if we cut off H_1 at $\{D_1,\ D_3\}$, then we get a connected cut diagram.

If we apply the algorithm in theorem 1 to $G(m, l) \cup G(l, m)$, then we can get connected genus 3 cut diagrams $G_1(m, l) \cup G_1(l, m)$ in Fig. 19. The construction of this is left to the reader.



$G_1(m, l)$ $G_1(l, m)$

Fig. 19

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