A NEW PRESENTATION OF THE FUNDAMENTAL GROUP ASSOCIATED WITH THE HEEGAARD DIAGRAM

ヘーガード図から得られる新しい基本群の表示

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§1. Introduction.

In this paper we will give a new presentation of the fundamental group associated with the subdivision of the Heegaard diagram. A subdivision G(m, l) of the Heegaard diagram $(U; m_1, \cdots, m_n, l_1, \cdots, l_n)$ is defined from cutting off the genus n Heegaard handlebody U at the complete system of the meridian disks D_i $(i=1,\cdots,n)$ of U. So each l_i of the longitude system $\{l_1, \cdots, l_n\}$ is decomposed into 1-cells. A new presentation of the fundamental group is represented of these 1-cells. This presentation is based on a manner of the caluculation of the fundamental group from a polyhedral presentation of a 3-manifold in [1]. A polyhedral presentation will also be called as a polygram in [3]. Our results are given in §3, theorem A, B and their corollaries. §2 are preliminaries; we will give definitions of a subdivision of the Heegaard diagram and the polygram. And we will state a relation that a pair of subdivisions $G(m, l) \cup G(l, m)$ of the Heegaard diagrams (U; m, l) and (V; l, m) becomes a same structure as a polygram without connectedness. In §4, we will prove theorems. In §5, we will give examples; Poincaré space, dodecahedron space and L(7, 2) #L(7, 4). Where the Poincaré space and the dodecahedron space are homeomorphic and they are generally called the homology 3-shere. But the constructions of them are different. Hence we will use the different names about them. The Poincaré space is constructed by H. Poincaré in 1904 as a genus 2 Heegaard diagram of a homology 3-sphere which is not homeomorphic to the 3-sphere S^3. A polygram of the dodecahedron space is given in [1], p224 fig. 112. The caluculations of the fundamental group from this polygram are given in p.224. We can also obtain a genus 2 Heegaard diagram of the dodecahedron space from the relations (I) and (II). But two presentations of the Heegaard diagrams are different. Hence two presentations of the fundamental groups which are obtained from the meridian, longitude system of these two Heegaard diagrams are different. However these two different presentations are obtained at once from the only one presentation of the fundamental group from the G(m, l) of the Heegaard diagram of the dodecahedron space.

We work in piecewise linear category throughout this paper.

§2. Heegaard diagram and polygram.

Definition 1. Suppose U is a genus n handlebody. Let $\{D_1, \cdots, D_n\}$ be a set of pairwise disjoint properly embedded 2-disks in U. If the closure of $U - \{D_1 \cup \cdots \cup D_n\}$ is a 3-ball, then $\{D_1, \cdots, D_n\}$ is called a **complete system of meridian disks** of U and each disk D_i is called a **meridian disk** of U. $\{\partial D_1, \cdots, \partial D_n\}$ (∂D_1 means boundary of D_1) is called a **complete system of meridian curves** of ∂U (or U) and each curve ∂D_i is called a **meridian curve.**

Note that $\{\partial D_1, \cdots, \partial D_n\}$ cut ∂U into a 2-sphere with 2n holes.

Definition 2. A connected orientable closed 3-manifold M^3 is represented with a union of two handlebodies U, V in M^3, $M^3 = U \cup V$ such that $U \cap V = \partial U \cap \partial V = \partial U = \partial V$ is a orientable closed surface F of genus n. A triple (U, V, F) is called a **Heegaard splitting** of M^3 with genus n and U or V is called the **Heegaard handlebody**. F is called the **Heegaard surface.** Furthermore, M^3 is obtained by identifying two handlebodies U and V of the same genus along their surface ∂U and ∂V, that is, an identification map f: $\partial U \to \partial V$ is an orientation-reversing homeomorphism. (M^3; U, V, f) is also called a **Heegaard splitting** concerning f.

Hereafter, a closed 3-manifold M^3 denotes a connected orientable closed 3-manifold unless otherwise stated.

Definition 3. Suppose (U, V, F)(resp. (M^3; U, V, f)) is a genus n Heegaard splitting of M^3. Let $l = \{l_1, \cdots, l_n\} = \{\partial D_1', \cdots, \partial D_n'\}$ and $m = \{m_1, \cdots, m_n\} = \{\partial D_1, \cdots, \partial D_n\}$ be a complete system of the meridian curves of ∂V and ∂U, respectively. Then (U; m, l)(resp. (U; m, $f^{-1}(l)$)) or (V; l, m) (resp. (V; l, f(m))) is called a **genus n Heegaard diagram** associated with (U, V, F)(resp. (M^3; U, V, f)). $l = \{l_1, \cdots, l_n\}$ (resp. $\{f^{-1}(l)\}$) of (U; m, l) (resp. (U; m, $f^{-1}(l)$)) is called the **longitude system.** And $m = \{m_1, \cdots, m_n\}$ of (U; m, l) or (U; m, $f^{-1}(l)$) is called the **meridian system.** Similarly, meridian, longitude system of (V; l, m) or (V; l, f(m)) are defined.

In (M^3; U, V, f), by replacing afresh $f^{-1}(V)$ with V, one can consider f^{-1} as a identification map. Hence $\{f^{-1}(l)\}$ is regarded as the longitude system $l = \{l_1, \cdots, l_n\}$ of (U; m, l).

Let (U; $m_1, \cdots, m_n, l_1, \cdots, l_n$) be a genus n Heegaard diagram associated with (U, V, F). We may assume $(m_1 \cup \cdots \cup m_n) \cap (l_1 \cup \cdots \cup l_n)$ consists of at most finite points (by an argument of general position).

Definition 4. The number of finite points of $(m_1 \cup \cdots \cup m_n) \cap (l_1 \cup \cdots \cup l_n)$ is called the **cross point number** with the genus n Heegaard diagram (U; m, l).

Now we will define a subdivision of a Heegaard diagram.

Let $\{D_1, \cdots, D_n\}$ and $\{D_1', \cdots, D_n'\}$ be complete system of meridian disks of genus n handlebodies U and V, respectively. Let $m = \{m_1, \cdots, m_n\} = \{\partial D_1, \cdots, \partial D_n\}$ and $l = \{l_1, \cdots, l_n\} = \{\partial D_1', \cdots, \partial D_n'\}$ be complete system of the meridian curves of ∂ U and ∂ V, respectively. Let (U; m, l) and (V; l, m) be genus n Heegaard diagrams of (U, V, F). Suppose each 1-sphere l_i, m_i $(i=1,\cdots,n)$ in F is oriented. The meridian system $\{m_1, \cdots, m_n\}$ in ∂ U decompose each 1-sphere l_i into 1-cells. We put labels $l_{i1}, l_{i2}, \cdots, l_{ik_i}$, in these order, to these 1-cells in l_i according to the orientation of the l_i such that $l_i = l_{i1} \cup l_{i2} \cup \cdots \cup l_{ik_i}$ (l_{i1}, the first 1-cell of l_i, is arbitrarily chosen from 1-cells in l_i). We may assume each label l_{ij} is oriented with the same orientation as l_i. In the opposite, the longitude system $\{l_1, \cdots, l_n\}$ in ∂ U decompose each 1-sphere m_j into 1-cells. We put labels $m_{j1}, m_{j2}, \cdots, m_{jl_j}$ in these order, to these 1-cells in m_j according to the orientation of m_j such that $m_j = m_{j1} \cup m_{j2} \cup \cdots \cup m_{jl_j}$. Let each label m_{ji} be oriented with the same orientation of m_j.

We cut off U at D_j $(j=1,\cdots,n)$, the complete system of meridian disks of U. Then we can get a 3-ball B_U^3; ∂B_U^3 is a 2-sphere S_U^2. In S_U^2, there are n pairs $\{D_j^+, D_j^-\}$ by cutting off U at D_j. Since ∂D_j^+ and ∂D_j^- are decomposed by same 1-cells in $\partial D_j = m_j$, they have oriented labels m_{j1}, $m_{j2}, \cdots, m_{jl_j}$ in common. Hence wa have a planar 3-regular graph in S_U^2 which is consisted of oriented labelled 1-cells such that

$\partial D_j^+ = m_j = m_{j1} \cup m_{j2} \cup \cdots \cup m_{jl_j}$, $\quad \partial D_j^- = m_j = m_{j1} \cup m_{j2} \cup \cdots \cup m_{jl_j}$

and $\partial D_i' = l_i = l_{i1} \cup l_{i2} \cup \cdots \cup l_{ik_i}$ $(i,\ j=1,\cdots,n)$.

Definition 5. A planar 3-regular graph

$$\{\partial D_j^+ = m_j = m_{j1} \cup \cdots \cup m_{jl_j},\ \partial D_j^- = m_j = m_{j1} \cup \cdots \cup m_{jl_j},$$
$$\partial D_i' = l_i = l_{i1} \cup \cdots \cup l_{ik_i}\} \quad (i,\ j=1,\cdots,n)$$

in S_U^2 is called a **subdivision associated with the Heegaard diagram (U; m, l)** and is described as **G(m, l)**. Similarly, a subdivision G(l, m) of (V; l, m) is defined and its expression is as follows;

$$G(l,\ m) = \{\ \partial D_i'^+ = l_i = l_{i1} \cup \cdots \cup l_{ik_i},\ \partial D_i'^- = l_i = l_{i1} \cup \cdots \cup l_{ik_i},$$
$$\partial D_j = m_j = m_{j1} \cup \cdots \cup m_{jl_j}\} \quad (i,\ j=1,\cdots,n).$$

A pair **G(m, l) $\cup$ G(l, m)** is called a **pair of subdivisions** of (U; m, l) and (V; l, m).

Sometimes, G(m, l) or G(l, m) will also be called as a Heegaard diagram.

Example 1. In the figure 1, there are genus 2 Heegaard diagrams (U; m_1, m_2, l_1, l_2), (V; l_1, l_2, m_1, m_2) of (U, V, F) of the 3-sphere S^3. G(m, l) and G(l, m) are subdivisions of (U; m_1, m_2, l_1, l_2) and (V; l_1, l_2, m_1, m_2), respectively. Then G(m, l) $\cup$ G(l, m) is a pair of their Heegaard diagrams.

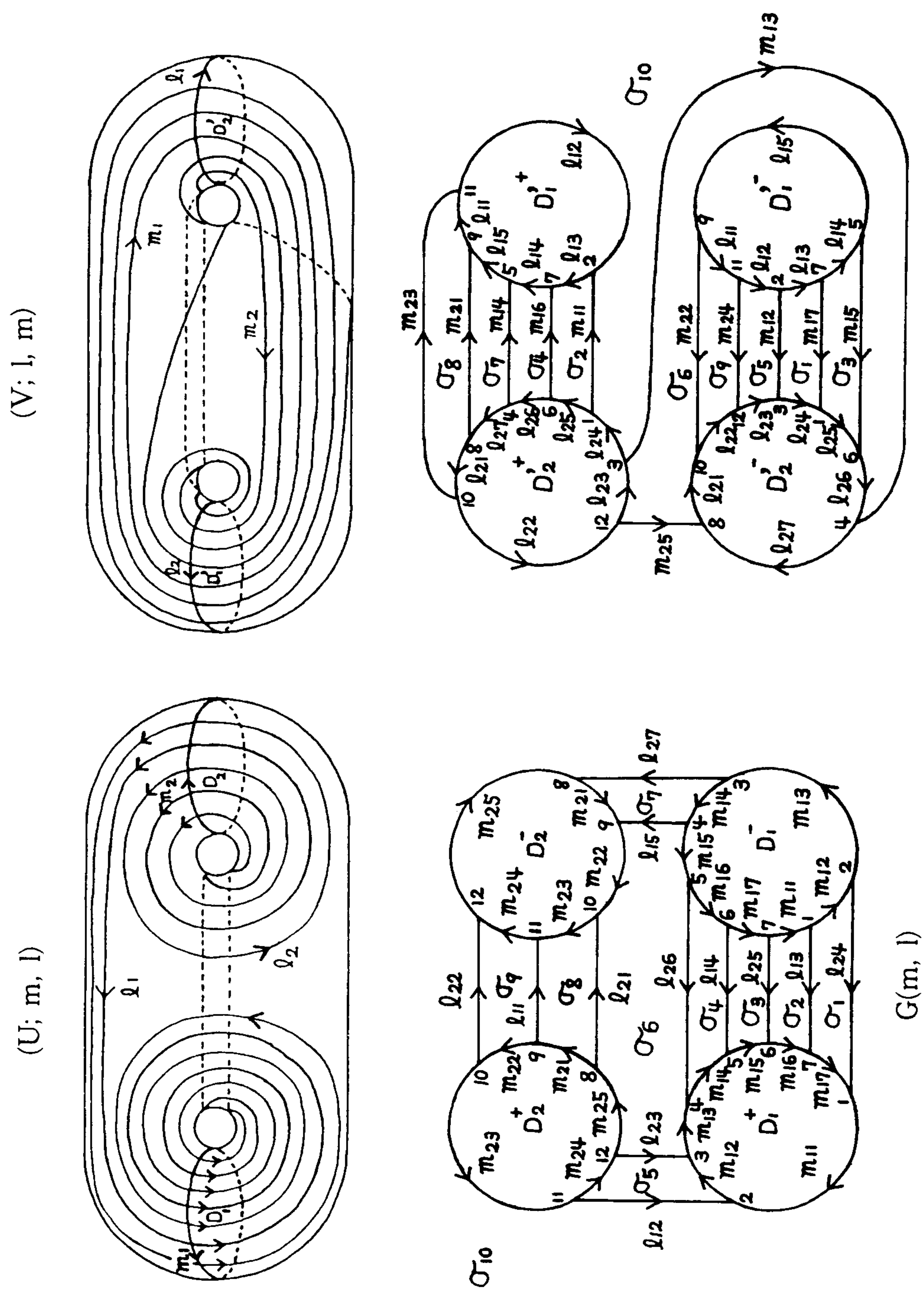


Figure 1

Definition 6. Let $(U;\ m,\ l)=(U;\ m_1,\ \cdots,\ m_n,\ l_1,\ \cdots,\ l_n)$ be a genus $n(\geqq 2)$ Heegaard diagram of $(U,\ V,\ F)$ and let the subdivision $G(m,\ l)$ be the same presentation as in definition 5. We choose $n-1$ pieces of l_{ij} of $G(m,\ l)$ and let these 1-cells be $\{L_1,\ \cdots,\ L_{n-1}\}$. If $(L_1 \cup \cdots \cup L_{n-1}) \cup (m_1 \cup \cdots \cup m_n)$ is a connected graph, then $(U;\ m,\ l)$ is called **connected** for the meridian system m. If we cannot choose any such 1-cells from $G(m,\ l)$, then $(U;\ m,\ l)$ is called **disconnected** for m. The connected orientable closed 3-manifolds which have genus 1 Heegaard diagram are 3-sphere, lens spaces $L(p,\ q)$ and $S^2 \times S^1$. We define that each genus 1 Heegaard diagram of the 3-sphere and the lens spaces is connected and genus 1 Heegaard diagram of the $S^2 \times S^1$ is disconnected for the meridian curve m_1 of ∂U.

Definition 7. Let $(U;\ m,\ l)=(U;\ m_1,\ \cdots,\ m_n,\ l_1,\ \cdots,\ l_n)$ be a genus $n(\geqq 2)$ Heegaard diagram of $(U,\ V,\ F)$. Let $\{\tilde{m}_1,\ \cdots,\ \tilde{m}_k\}$ and $\{\tilde{l}_1,\ \cdots,\ \tilde{l}_l\}$ be subset of the meridian system m and the longitude system l, respectively. If $(\tilde{m}_1 \cup \cdots \cup \tilde{m}_k) \cup (\tilde{l}_1 \cup \cdots \cup \tilde{l}_l)$ is a connected graph, then this connected graph is called a **connected component** of $(U;\ m,\ l)$.

Definition 8. Let $G(m,\ l)$ be the same presentation as in definition 5. If the closures of connected components of
$S_U^2 - |G(m,\ l)| - \bigcup_{j=1}^{n} (\mathrm{Int}\ D_j^+ \cup \mathrm{Int}\ D_j^-)$ consists of 2-cells, then $G(m,\ l)$ is called **connected** for the meridian system m. Where $|G(m,\ l)|$ denotes the underlying space of $G(m,\ l)$ in S_U^2. If one cannot get a connected graph from $G(m,\ l)$, then $G(m,\ l)$ is called **disconnected** for m.

Let $G(m,\ l)$ and $G(l,\ m)$ be the same presentations as in definition 5. Let each P_j $(j=1,\cdots,n)$ be the center point of the meridian disk D_j of U and let P_j^+ and P_j^- be the center points of $D_j^+(\subset S_U^2)$ and $D_j^-(\subset S_U^2)$, respectively. In the S_U^2, if we deform G(m ,l) such that D_j^-, D_j contract to P_j^+, P_j, respectively, then we can get a new planar graph on a 2-sphere S^2. This planar graph consists of vertices $\{P_i^+, P_i\}$ $(i=1,\cdots,n)$ and edges $\{l_{i1}, \cdots, l_{ik_i}\}$ $(i=1,\cdots,n)$. And $\{\partial l_{i1}, \cdots, \partial l_{ik_i}\}$ $(i=1,\cdots,n)$ are vertices $\{P_i^+, P_i\}$. We put a same label P_i on these two points which are labelled as P_i^+, P_i, respectively.

Definition 9. Such graph is called a **Whitehead graph** (or simply called **W-graph**) of the G(m, l) and is denoted by **WG(l)** (or **WG($l_1, \cdots, l_n$)**). We express WG(l) by $\{l_{i1}, \cdots, l_{ik_i}, P_i\}$ $(i=1,\cdots,n)$. Similarly, WG(m) is defined from the G(l, m) and its expression is WG(m)$=\{m_{j1}, \cdots, m_{jl_j}, Q_j\}$ $(j=1,\cdots,n)$ where Q_j is the center point of the meridian disk D_j' of V. WG(m) is called the **dual graph** of WG(l).

Definition 10. WG(l) is called **connected** (resp. **disconnected**) if G(m, l) is connected (resp. disconnected).

Example 3. Figure 2 is a connected genus 4 Heegaard diagram of $S^2 \times S^1$ and its disconnected subdivision G(m, l). And W-graph WG(l) of G(m, l) is also disconnected.

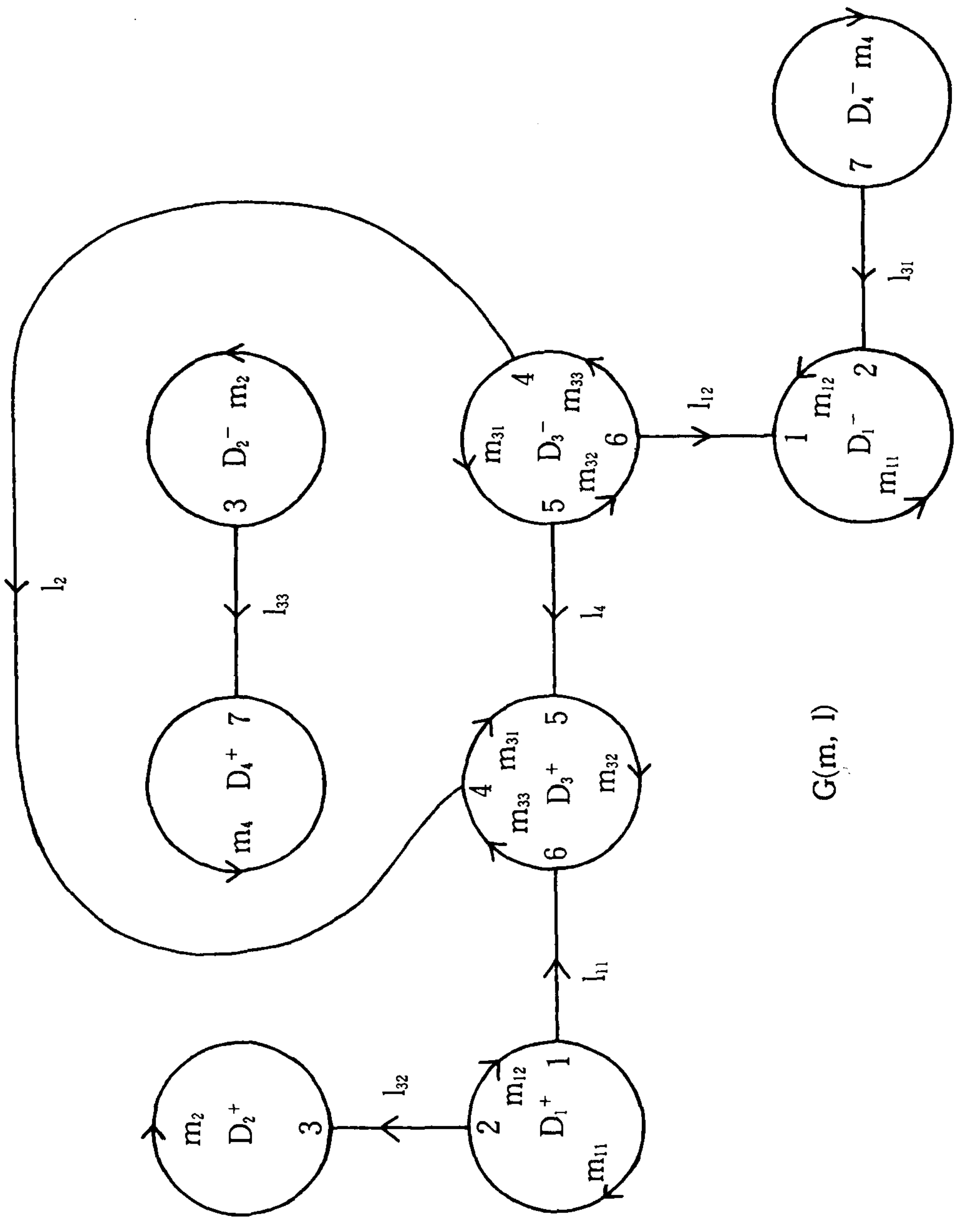

D_1^+
D_2^+
D_3^+
D_4^+
D_1^-
D_2^-
D_3^-
D_4^-
m_{11}
m_{12}
m_2
m_{31}
m_{32}
m_{33}
m_4
l_{11}
l_{12}
l_2
l_{31}
l_{32}
l_{33}
l_4
G(m, l)

Figure 2

It is equivalent that G(m, l) is connected and its W-graph WG(l) is connected. It is clear that if G(m,l) of (U; m, l) is connected, then (U; m, l) also becomes connected. But from example 3, the reverse of this does not hold generally. If Heegaard genus = 1, then it is equivalent that (U; m_1, l_1) is connected and its subdivision G(m_1, l_1) is connected.

Definition 11. Let G=(V; E) be a connected graph on a 2-sphere S^2, where V, E denotes the set of vertices, edges of G, respectively. Let F be the set of closures of connected components of $S^2 - |G|$. Then each element of V, E or F is called as a **0-**, **1-**, or **2-cell**, respectively. Hence $K=V\cup E\cup F$ becomes naturally oriented 2-dimensional cell complex if we designate an orientation for each i-cell σ in K. Such K is called a **P-complex**.

Definition 12. Let G=(V; E) be a connected graph on a 2-sphere S^2 and $K=V\cup E\cup F$ be the P-complex of G. Suppose K satisfies the following conditions (1), (2) or (3).

(1) $\{v_i', v_i'', \cdots\}$ $(i=1,\cdots,\alpha^0)$ are 0-cells of V such that a same label v_i is put on these 0-cells.

$\{e_i', e_i'', \cdots\}$ $(i=1,\cdots,\alpha^1)$ are 1-cells of E such that a same label e_i is put on these 1-cells.

$\{f_i', f_i''\}$ $(i=1,\cdots,\alpha^2)$ are 2-cells of F such that a same label f_i is put on these 2-cells.

(2) For each 2-cells f_i', f_i'' of F which a same label f_i is put on, the orientation of f_i' is an opposite to the orientation of f_i''. f_i' and f_i'' satisfy that $\partial f_i' \cap \partial f_i'' \neq \phi$ or $\partial f_i' \cap \partial f_i'' = \phi$ and that the number of 1-cells in $\partial f_i'$ equal to the number of 1-cells in $\partial f_i''$.

(3) Gluing together f_i' and f_i'' and gluing together 0-, 1-cells in $\partial f_i'$ and

$\partial f_i''$ are compatible including the labels of 0-, 1-cells and the orientations of 1-cells.

Let f be a map such that f identifies 0-, 1- or 2-cells with same labels, respectively. Then $\Lambda=(G, f)$ is called a **polygram**.

Note that a polygram is defined as a **polyhedral representation** in [1]. And precise definition of a polygram is given in [3].

Proposition. *Let $(U; m, l)$ and $(V; l, m)$ be genus $n(\geqq 1)$ Heegaard diagrams of (U, V, F). Let $G(m, l) \cup G(l, m)$ be a pair of the subdivisions of $(U; m, l)$ and $(V; l, m)$. And let $G(m, l)$ be connected. Then $G(m, l) \cup G(l, m)$ satisfies the conditions of a polygram without connectedness.* (M. YAMASHITA [2])

Proof. Let G(m, l) and G(l, m) be the same presentations as in definition 5, respectively. Let $\sigma_1,\cdots,\sigma_p$ be 2-cells that is the closures of connected components of $S_U^2 - |G(m, l)| - \bigcup_{j=1}^{n} (\mathrm{Int}\ D_j^+ \cup \mathrm{Int}\ D_j^-)$. And let the name σ_i of 2-cell be the label of its. Since $\partial U = \partial V (=F)$, each 2-cell of the closures of connected components of $S_V^2 - |G(l, m)| - \bigcup_{j=1}^{n} (\mathrm{Int}\ D_j'^+ \cup \mathrm{Int}\ D_j'^-)$ is put on the same label σ_i of the 2-cell in S_U^2 (see figure 1). $m \cap l = (m_1 \cup \cdots \cup m_n) \cap (l_1 \cup \cdots \cup l_n)$ consists of cross points of (U; m, l) or (V; l, m). Let X be one of a point of them. Since there are two 1-cells which have a same label m_{ji} in G(m, l), if $X \subset \partial m_{ji}$, then in G(m, l), there exist two 0-cells which have a same label X, the name of the 0-cell. In G(l, m), there is a 1-cell m_{ji}. Since there exist four disks $\{D_k'^+, D_k'^-, D_l'^+, D_l'^-\}$ in S_V^2 such that $\partial m_{ji} \subset \{\partial D_k'^+, \partial D_k'^-, \partial D_l'^+, \partial D_l'^-\}$, if $X \subset D_k'^+$, then $X \subset D_k'^-$. We put the same label D_i (resp. D_j') on the two disks D_i^+, D_i^- (resp. $D_j'^+$, $D_j'^-$) in S_U^2 (resp. S_V^2) $(i, j=1,\cdots,n)$. Hence M^3 is obtained by identifying four 0-cells apiece with a same label, three 1-cells apiece with a same label or two 2-cells apiece with a same label in S_U^2 and S_V^2, the boundaries of B_U^3, B_V^3, respectively. Let f be the identification map above same labels. Then $(G(m, l) \cup G(l, m), f)$ satisfies the conditions of a polygram without connectedness. Q. E. D.

§3. Presentation of the fundamental group of the Heegaard diagram.

Let (U; m, l) be a genus $n(\geqq 1)$ Heegaard diagram of (U, V, F) and G(m, l) be the subdivision of (U; m, l). Let G(m, l) be connected and G(m, l) be the same presentation as in definition 5. Since (U; m, l) becomes connected, one can choose $n-1$ pieces of 1-cell from $\{l_{ij}\}$ of G(m, l) such that these 1-cells connect the meridian curves $\{m_1, \cdots, m_n\}$ of (U; m, l). Let these 1-cells be $\{L_1, L_2, \cdots, L_{n-1}\}$. Let $\sigma_1, \cdots, \sigma_p$ be 2-cells that is the closures of connected components of $S_U^2 - |G(m, l)| - \bigcup_{j=1}^{n} (\mathrm{Int}\, D_j^+ \cup \mathrm{Int}\, D_j^-)$. From now on we will use each label l_{ij} of 1-cells together with as a symbol of the generators of a fundamental group.

Theorem A. *Let (U; m, l) be a genus $n(\geqq 2)$ Heegaard diagram of (U, V, F) of M^3. Let the subdivision G(m, l) of (U; m, l) be connected for the meridian system m. Then $l_{i1}, \cdots, l_{ik_i}$ $(i=1,\cdots,n)$ become generators of a fundamental group of M^3. Relations are as follows;*

(1) $L_1=1, \cdots, L_{n-1}=1$

(2) $r_i=1$ *where r_i is a word which is obtained by reading the labels $\{l_{ij}\}$ of 1-cells in $\partial\sigma_i$ continuously, omitting m_{ij} as l_{ij} (resp. l_{ij}^{-1}) if the orientation of 1-cell is same (resp. opposite) as the orientation for*

running around the $\partial\sigma_i$ $(i=1,\cdots,p)$.

(3) $l_{i1} \cdots l_{ik_i}=1$ $(i=1,\cdots,n)$

In the relations (2), note that we may start reading from an any 1-cell in $\partial\sigma_i$ because the word r_i becomes a cyclic word by joining the beginning and the end of r_i and preserving the sequential order of letters in r_i. Hence r_i is uniquely determined.

We will give a example for theorem A before its proof.

Example 4. A presentation of a fundamental group of S^3 which is obtained from G(m, l) in figure 1 is as follows; here we put l_{23} at L_1 as the relator given in (1) of theorem A.

$$\pi_1(S^3)=\left\langle \begin{array}{l|l} l_{11},l_{12},l_{13},l_{14},l_{15} & L_1=l_{23}=1,\ l_{24}l_{13}^{-1}=1,\ l_{13}l_{25}^{-1}=1,\ l_{25}l_{14}^{-1}=1 \\ l_{21},l_{22},l_{23},l_{24},l_{25} & l_{14}l_{26}^{-1}=1,\ l_{23}l_{12}^{-1}=1,\ l_{15}l_{21}^{-1}l_{23}l_{26}^{-1}=1 \\ l_{26},l_{27} & l_{15}l_{27}^{-1}=1,\ l_{21}l_{11}^{-1}=1,\ l_{11}l_{22}^{-1}=1 \\ & l_{12}l_{24}^{-1}l_{27}l_{22}^{-1}=1,\ l_{11}l_{12}l_{13}l_{14}l_{15}=1 \\ & l_{21}l_{22}l_{23}l_{24}l_{25}l_{26}l_{27}=1 \end{array}\right\rangle$$

We denote a presentation of the fundamental group given in theorem A as follows;

$$\pi_1(M^3)=<\ l_{ij}\ \mid\ L_1=1,\ \cdots,\ L_{n-1}=1,\ r_1=1,\ \cdots,\ r_p=1,\ l_{i1}\cdots l_{ik_i}=1\ >\cdots(\mathrm{A})$$
$$(i=1,\cdots,n)$$

As to this presentation, there are three same labels apiece in the relators $\{r_1,\ \cdots,\ r_p,\ l_{i1}\cdots l_{ik_i}\}$ $(i=1,\cdots,n)$ without orientations of 1-cells.

Corollary 1. *Let* $(U;\, m_1,\, l_1)$ *be a genus* 1 *connected Heegaard diagram of* $(U,\, V,\, F)$ *of* M^3 *and* $G(m_1,\, l_1)$ *be the subdivision of* $(U;\, m_1,\, l_1)$. *Then* $l_{11},\, \cdots,\, l_{1k}$ *become generators of the fundamental group. Relations are* $r_i=1$ $(i=1,\cdots,p)$ *and* $l_{11}\cdots l_{1k}=1$.

Example 5. Figure 3 gives genus 1 connected Heegaard diagrams $(U;\, m_1,\, l_1)$ and $(V;\, l_1,\, m_1)$ of the lens space $L(7,\, 2)$ and its subdivisions $G(m_1,\, l_1)$ and $G(l_1,\, m_1)$, respectively. A presentation of the fundamental group from $G(m_1,\, l_1)$ is as follows;

$$\pi_1(L(7,\,2))=\left\langle \begin{array}{l} l_{11},\, l_{12},\, l_{13},\, l_{14} \\ l_{15},\, l_{16},\, l_{17} \\ \ \end{array} \;\middle|\; \begin{array}{l} l_{11}l_{14}^{-1}=1,\ l_{14}l_{17}^{-1}=1,\ l_{17}l_{13}^{-1}=1 \\ l_{13}l_{16}^{-1}=1,\ l_{16}l_{12}^{-1}=1,\ l_{12}l_{15}^{-1}=1 \\ l_{11}l_{15}^{-1}=1,\ l_{11}l_{12}l_{13}l_{14}l_{15}l_{16}l_{17}=1 \end{array} \right\rangle$$

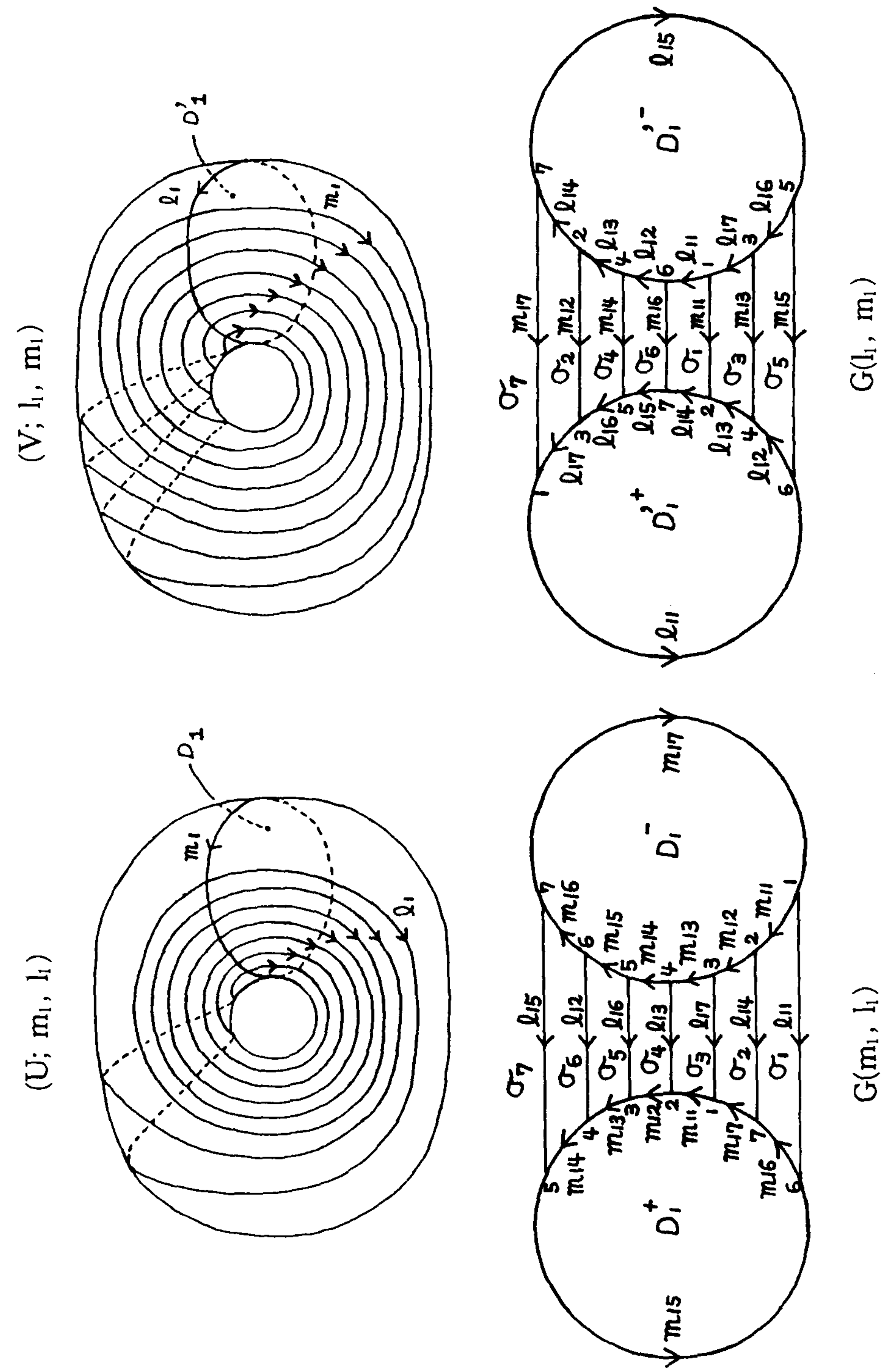


Figure 3

Corollary 2. *Let $G(l, m)$ be the pair of $G(m, l)$. From $G(l, m)$, one can obtain a fundamental group which is presented by* $\{m_{j1}, \cdots, m_{jl_j}\}$ $(j=1,\cdots,n)$.

Next we will give a presentation of a fundamental group from G(m, l) which is disconnected.

Let (U; m, l) be a genus $n(\geqq 1)$ Heegaard diagram of (U, V, F) of M^3 and G(m, l) be a disconnected subdivision of (U; m, l). Let G(m, l) be the same presentation as in definition 5. Let $\xi_1, \cdots, \xi_g (g \geqq 1)$ be not 2-cells in the closures of connected components of $S_U^2 - |G(m, l)| - \bigcup_{j=1}^{n} (\mathrm{Int}\ D_j^+ \cup \mathrm{Int}\ D_j^-)$. We will take 1-cells $X_{i1}, \cdots, X_{it_i}$ in ξ_i as follows;

(1) Let each 1-cell X_{ij} be oriented.

(2) $\mathrm{Int}\ X_{ij} \subset \mathrm{Int}\ \xi_i$ and let ∂X_{ij} be two points $\{Q_i, Q_j\}$ $(Q_i \neq Q_j)$. Then
$Q_i \subset l_{ij} \subset \partial \xi_i$ and $Q_j \subset l_{kl} \subset \partial \xi_i$ $(l_{ij} \neq l_{kl})$
or $Q_i \subset l_{ij} \subset \partial \xi_i$ and $Q_j \subset m_{kl} \subset \partial \xi_i$
or $Q_i \subset m_{ij} \subset \partial \xi_i$ and $Q_j \subset m_{kl} \subset \partial \xi_i$ $(m_{ij} \neq m_{kl})$.

(3) The closures of connected components of $\xi_i - (\mathrm{Int}\ X_{i1} \cup \cdots \cup \mathrm{Int}\ X_{it_i})$ become 2-cells.

Then $G(m, l) \cup (\bigcup_{i=1}^{g} (X_{i1} \cup \cdots \cup X_{it_i}))$ becomes a connected graph. Let $(\bigcup_{i=1}^{g} \{\partial X_{i1}, \cdots, \partial X_{it_i}\}) \cap (\mathrm{Int}\ l_{ij})$ be points $\{P_{ij1}, \cdots, P_{ijr_i-1}\}$. Then 1-cell l_{ij} is decomposed into 1-cells $l_{ij1}, \cdots, l_{ijr_i}$ by these points such that $l_{ij} = l_{ij1} \cup \cdots \cup l_{ijr_i}$. Let each 1-cell l_{ijk} be the same orientation as l_{ij}. Similarly 1-cell m_{ji} is decomposed into 1-cells $m_{ji1}, \cdots, m_{jis_j}$. In the G(m, l), if such l_{ij}, m_{ji} arise, then we change from l_{ij}, m_{ji} into the decomposed 1-cells $\{l_{ij1}, \cdots, l_{ijr_i}\}$, $\{m_{ji1}, \cdots, m_{jis_j}\}$, respectively. We denote such constructed graph from G(m, l) as G'(m, l). G'(m, l) is presented as follows;

$$G'(m,\ l)=\ \{\ \partial D_j^+ = m_{j1} \cup \cdots \cup m_{jl_j},\ \partial D_j^- = m_{j1} \cup \cdots \cup m_{jl_j},\ \partial D_i' = l_{i1} \cup \cdots \cup l_{ik_i},\ l_{uv} = l_{uv1} \cup \cdots \cup l_{uvr_u},\ m_{xy} = m_{xy1} \cup \cdots \cup m_{xys_x}\ \}$$
$$(i=1,\cdots,n-\alpha,\ j=1,\cdots,n-\beta,\ u=1,\cdots,\alpha,\ x=1,\cdots,\beta)$$

Note that if $\bigcup_{i=1}^{g}\ \{\ \partial X_{i1},\ \cdots,\ \partial X_{it_i}\} \subset \{m_1 \cup \cdots \cup m_n\} \cap \{l_1 \cup \cdots \cup l_n\}$ then $G'(m,\ l)=G(m,\ l)$.

Since $G'(m,\ l) \cup (\ \bigcup_{i=1}^{g}\ (X_{i1} \cup \cdots \cup X_{it_i}))$ becomes connected from the condition (3), we can construct a path from 1-cells of $G'(m,\ l) \cup (\ \bigcup_{i=1}^{g}\ (X_{i1} \cup \cdots \cup X_{it_i}))$ such that a path, in other words a tree of the graph theory, connect each m_i of $m(=m_1 \cup \cdots \cup m_n)$ and each point of $\bigcup_{i=1}^{g}\ \{\ \partial X_{i1},\ \cdots,\ \partial X_{it_i}\}$. We construct such a path as follows;

(a) In the case that Heegaard diagram (U; m, l) is connected.

Let such a path be $(L_1 \cup \cdots \cup L_{n-1})\ \cup\ (\cup (l_{ij1}' \cup \cdots \cup l_{ijr_i}'))$. Where $\{L_1,\ \cdots,\ L_{n-1}\}$ are chosen from 1-cells $\bigcup_{i=1}^{n}\ \{l_{i1},\ \cdots,\ l_{ik_i}\}\ (l_i=l_{i1}\cdots l_{ik_i})$ and they connect each m_i of m. However if $L_i=l_{ij}=l_{ij1} \cup \cdots \cup l_{ijr_i}$, then we must change from 1-cell L_i into 1-cells $\{l_{ij1},\ \cdots,\ l_{ijr_i}\}$ in the path. If $l_{ij} \neq L_i$ and $l_{ij}=l_{ij1} \cup \cdots \cup l_{ijr_i}$ then $\{l_{ij1}' \cup \cdots \cup l_{ijr_i}'\}$ are chosen from $\{l_{ij1},\ \cdots,\ l_{ijr_i}\}$ such that $l_{ij1}' \cup \cdots \cup l_{ijr_i}'$ reach to the division points of l_{ij}.

(b) In the case that Heegaard diagram (U; m, l) is disconnected.

Let such a path be

$(L_1 \cup \cdots \cup L_h) \cup (\cup (l_{ij1}' \cup \cdots \cup l_{ijr_i}')) \cup (\ \bigcup_{i=1}^{g} (X_{i1} \cup \cdots \cup X_{it_i}))\ (0 \leqq h < n-1)$.

Where $\{L_1,\ \cdots,\ L_h\}$ are chosen from 1-cells $\bigcup_{i=1}^{n}\ \{l_{i1},\ \cdots,\ l_{ik_i}\}$ $(l_i=l_{i1}\cdots l_{ik_i})$ as follows;

Let $C_1,\ \cdots,\ C_f (f \geqq 0,\ f \neq 1)$ be connected components of (U; m, l). Then

$\{L_1, \cdots, L_h\} = \{\ l_{ij}\ \mid\ \{l_{ij}\}$ are $n_i - 1$ pieces of 1-cell in C_i such that these 1-cells connect n_i pieces of m_i of m contained in C_i, $2 \leqq i \leqq f\ \}$

Note that if f=0 then $\{L_1, \cdots, L_h\} = \phi$ and if f=1 then its becomes the case (a).

However if $L_i = l_{ij} = l_{ij1} \cup \cdots \cup l_{ijr_i}$, then we must change from 1-cell L_i into 1-cells $\{l_{ij1}, \cdots, l_{ijr_i}\}$ in the path. If $l_{ij} \neq L_i$ and $l_{ij} = l_{ij1} \cup \cdots \cup l_{ijr_i}$ then $\{l_{ij1}' \cup \cdots \cup l_{ijr_i}'\}$ are chosen from $\{l_{ij1}, \cdots, l_{ijr_i}\}$ as in the case (a).

Let $\sigma_1, \cdots, \sigma_p$ be 2-cells that is the closures of connected components of $S_U^2 - \mid G'(m, l) \mid - \bigcup_{j=1}^{n} (\mathrm{Int}\ D_j^+ \cup \mathrm{Int}\ D_j^-) - \bigcup_{i=1}^{g} (X_{i1} \cup \cdots \cup X_{it_i})$.

Theorem B. *Let $(U;m, l)$ be a genus $n(\geqq 1)$ Heegaard diagram of (U, V, F) of M^3. Let the subdivision $G(m, l)$ of $(U; m, l)$ be disconnected for the meridian system m. Then a fundamental group of M^3 is represented by reading 1-cells of $G'(m, l) \cup (\bigcup_{i=1}^{g} (X_{i1} \cup \cdots \cup X_{it_i}))$ as follows;*

(a) *In the case that Heegaard diagram $(U; m, l)$ is connected.*

Generators are $\{l_{i1}, \cdots, l_{ik_i}\}$ $(i=1,\cdots,n-\alpha)$, $\{l_{uv1}, \cdots, l_{uvr_u}\}$ $(l_{uv} = l_{uv1} \cdots l_{uvr_u}$, $(u=1,\cdots,\alpha))$ *and* $\{X_{i1}, \cdots, X_{it_i}\}$ $(i=1,\cdots,g)$.

Relations are as follows;

(1) $L_1 = 1, \cdots, L_{n-1} = 1$

However if $L_i = l_{ij} = l_{ij1} \cup \cdots \cup l_{ijr_i}$, then we must change from $L_i = 1$ into $l_{ij1} = 1, \cdots, l_{ijr_i} = 1$.

(2) *If $l_{ij} \neq L_k$(for any k) and $l_{ij} = l_{ij1} \cup \cdots \cup l_{ijr_i}$ then $l_{ij1}' = 1, \cdots, l_{ijr_i}' = 1$.*

(3) *$r_i = 1$ where r_i is a word which is obtained by reading the labels*

$\{l_{ij}\}$ of 1-cells in $\partial\sigma_i$ continuously, omitting m_{ji}, m_{xyi} as l_{ij} (resp. l_{ij}^{-1}) if the orientation of 1-cell is same (resp. opposite) as the orientation for running around the $\partial\sigma_i$ $(i=1,\cdots,p)$.

However if $l_{ij}=l_{ij1}\cup\cdots\cup l_{ijr_i}$, then we must change from l_{ij} into $l_{ij1}\cdots l_{ijr_i}$. And we must read each decomposed 1-cell as l_{ijk} or l_{ijk}^{-1} for running around the $\partial\sigma_i$.

(4) $l_{i1}\cdots l_{ik_i}=1\,(i=1,\cdots,n)$

If $l_{ij}=l_{ij1}\cup\cdots\cup l_{ijr_i}$ then we must change from l_{ij} into $l_{ij1}\cdots l_{ijr_i}$.

(*b*) *In the case that Heegaard diagram $(U;\ m,\ l)$ is disconnected.*

Generators are the same contents as in (a).

Relations are as follows;

(1) $L_1=1,\ \cdots,\ L_h=1\,(0\leqq h<n-1)$

However if $L_i=l_{ij}=l_{ij1}\cup\cdots\cup l_{ijr_i}$, then we must change from $L_i=1$ into $l_{ij1}=1,\ \cdots,\ l_{ijr_i}=1$.

(2) *Relations are the same contents as* (2) *in* (*a*).

(3) *Relations are the same contents as* (3) *in* (*a*).

(4) *Relations are the same contents as* (4) *in* (*a*).

(5) $X_{i1}=1,\ \cdots,\ X_{it_i}=1\,(i=1,\cdots,g)$

We will denote these presentations of the fundamental groups as follows;

(*a*) $\pi_1(M^3)$

$$=\left\langle \begin{array}{l|l} l_{ij},\ l_{uvr} & L_1=1,\ \cdots,\ L_{n-1}=1,\ r_1=1,\ \cdots,\ r_p=1,\ l_{i1}\cdots l_{ik_i}=1 \\ X_{kl} & (L_u=l_{uv}=l_{uv1}\cdots l_{uvr_u}\rightarrow)\ l_{uv1}=1,\ \cdots,\ l_{uvr_u}=1 \\ & (l_{uv}=l_{uv1}\cdots l_{uvr_u}(l_{uv}\neq L_k)\rightarrow)\ l_{uv1}'=1,\ \cdots,\ l_{uvr_u}'=1 \end{array}\right\rangle \cdots\text{(B1)}$$

$$(i=1,\cdots,n-\alpha,\ u=1,\cdots,\alpha)$$

(b) $\pi_1(M^3)$

$$=\left\langle \begin{array}{l|l} l_{ij},\ l_{uvr} & L_1=1,\ \cdots,\ L_h=1,\ r_1=1,\ \cdots,\ r_p=1,\ l_{i1}\cdots l_{ik_i}=1 \\ X_{kl} & (L_u=l_{uv}=l_{uv1}\cdots l_{uvr_u}\rightarrow)\ l_{uv1}=1,\ \cdots,\ l_{uvr_u}=1 \\ & (l_{uv}=l_{uv1}\cdots l_{uvr_u}(l_{uv}\neq L_k)\rightarrow)\ l_{uv1}'=1,\ \cdots,\ l_{uvr_u}'=1 \\ & X_{k1}=1,\ \cdots,\ X_{kt_k}=1 \end{array}\right\rangle \cdots\text{(B2)}$$

$$(i=1,\cdots,n-\alpha,\ u=1,\cdots,\alpha,\ 0\leqq h<n-1,\ k=1,\cdots,g)$$

$$=\left\langle \begin{array}{l|l} l_{ij},\ l_{uvr} & L_1=1,\ \cdots,\ L_h=1,\ r_1=1,\ \cdots,\ r_p=1,\ l_{i1}\cdots l_{ik_i}=1 \\ & (L_u=l_{uv}=l_{uv1}\cdots l_{uvr_u}\rightarrow)\ l_{uv1}=1,\ \cdots,\ l_{uvr_u}=1 \\ & (l_{uv}=l_{uv1}\cdots l_{uvr_u}(l_{uv}\neq L_k)\rightarrow)\ l_{uv1}'=1,\ \cdots,\ l_{uvr_u}'=1 \end{array}\right\rangle$$

$$(i=1,\cdots,n-\alpha,\ u=1,\cdots,\alpha,\ 0\leqq h<n-1)$$

Example 6. Figure 4 is a disconnected genus 1 Heegaard diagram (U; m_1, l_1) of $S^2\times S^1$ and its subdivision G(m_1, l_1) and G'(m_1, l_1) $\cup X_1 \cup X_2$. The fundamental group by reading 1-cells of G'(m_1, l_1) $\cup X_1 \cup X_2$ is as follows;

$\pi_1(S^2\times S^1)=<l_{11},\ l_{12}\ |\ l_{11}\,l_{12}=1>\approx \mathbf{Z}$ ($\approx$ means isomorphism)

Example 7. Without two dotted lines X_{11} and X_{12}, figure 5 is a disconnected subdivision G(m, l) of a disconnected genus 4 Heegaard diagram (U; m, l) of S^3.

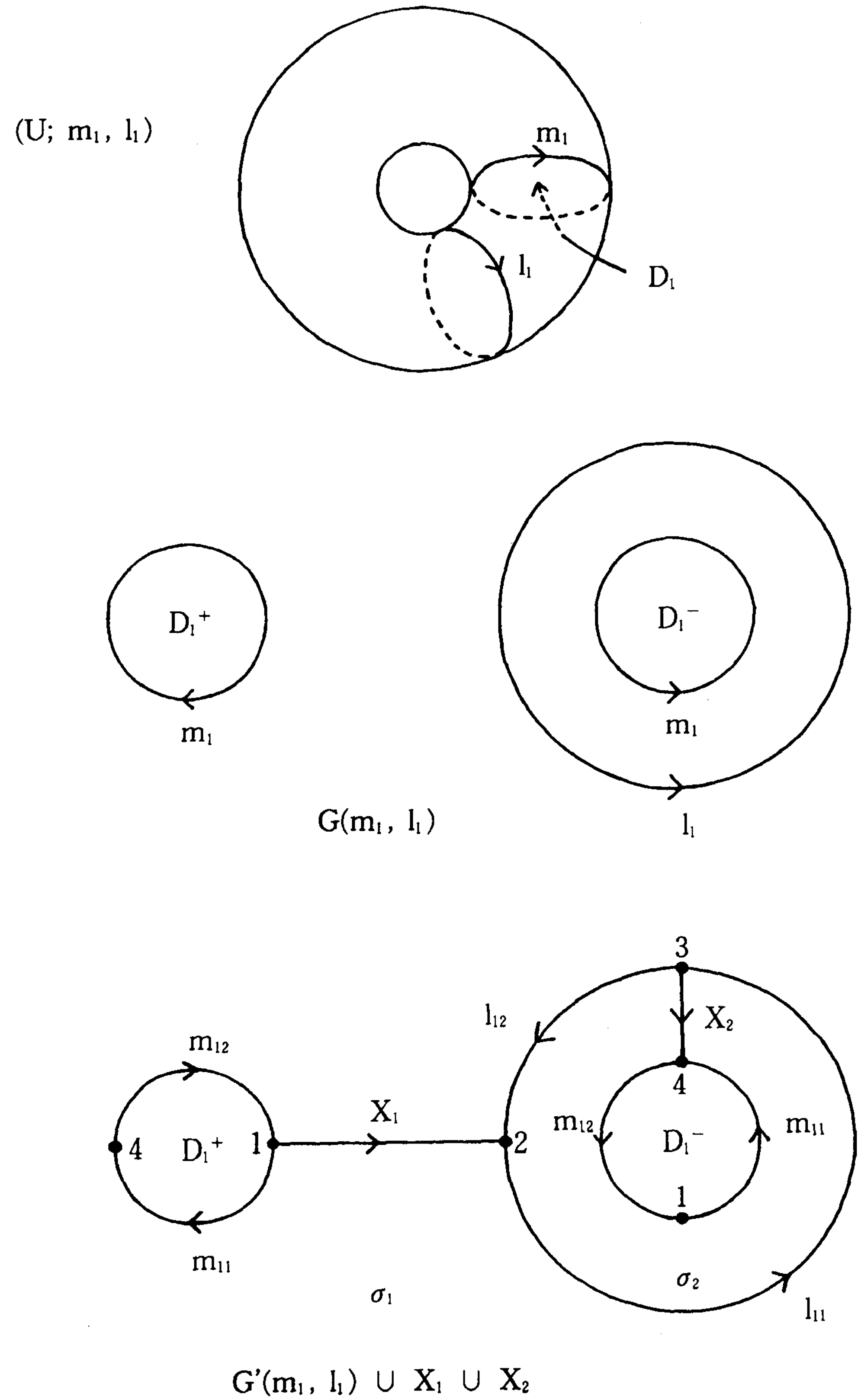
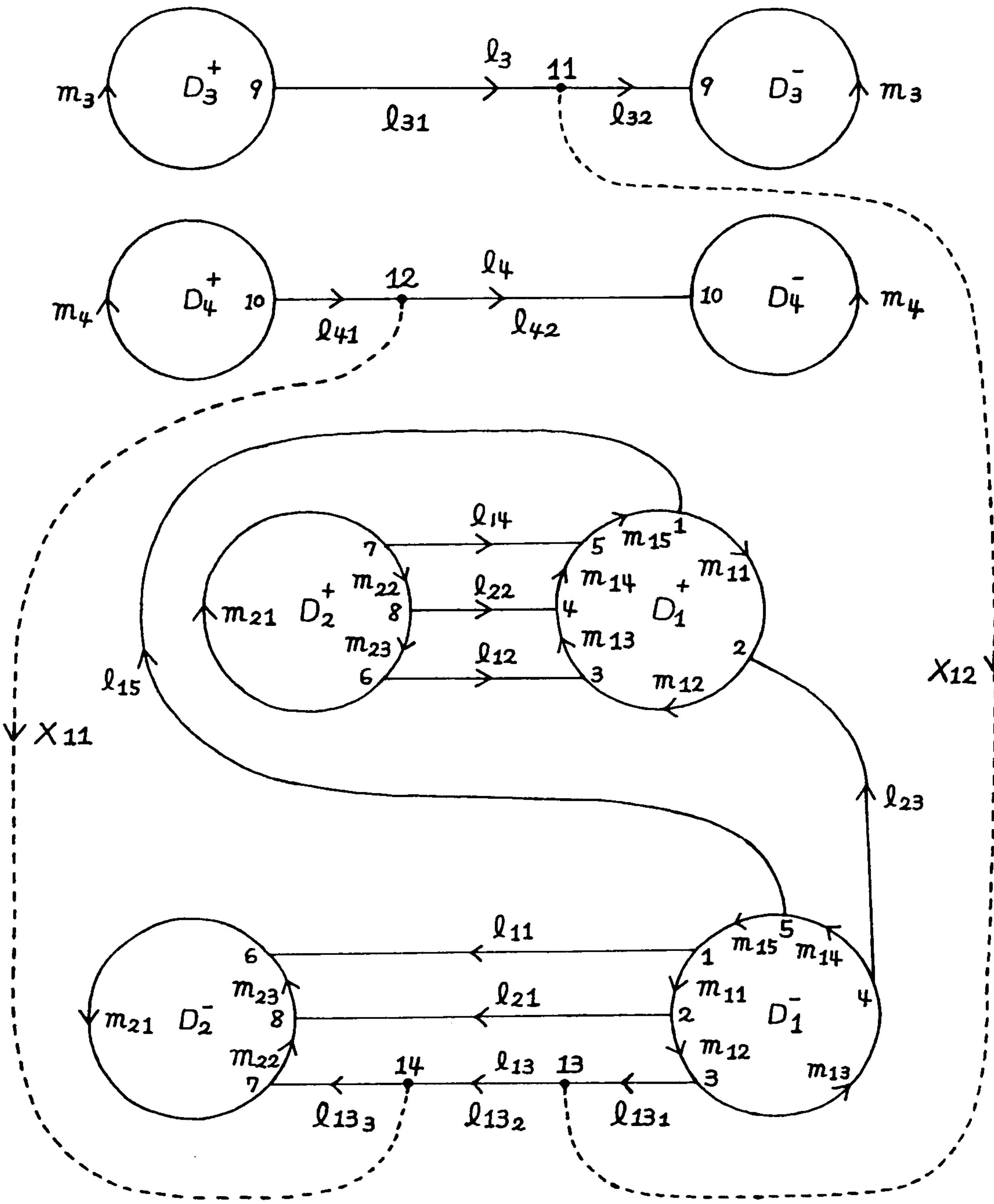

(U; m1, l1)
m1
l1
D1
D1+
m1
D1−
m1
l1
G(m1, l1)
3
X2
l12
m12
4
X1
4 D1+ 1
2
m12
D1−
m11
1
m11
σ1
σ2
l11
G'(m1, l1) ∪ X1 ∪ X2

Figure 4


m_3
D_3^+
9
l_3
11
l_{31}
l_{32}
9
D_3^-
m_3
m_4
D_4^+
10
12
l_4
l_{41}
l_{42}
10
D_4^-
m_4
l_{14}
7
m_{22}
m_{21}
D_2^+
8
m_{23}
6
l_{22}
l_{12}
5
m_{15}
1
m_{11}
m_{14}
4
D_1^+
m_{13}
2
3
m_{12}
l_{15}
X_{11}
X_{12}
l_{23}
l_{11}
6
m_{23}
m_{21}
D_2^-
8
m_{22}
7
l_{21}
14
l_{13}
13
l_{13_3}
l_{13_2}
l_{13_1}
5
1
m_{15}
m_{14}
m_{11}
D_1^-
4
2
m_{12}
m_{13}
3

G(m, l)

Figure 5

By adding X_{11}, X_{12} to G(m, l), we can obtain a connected graph G'(m, l) $\cup X_{11} \cup X_{12}$. There are 14 points in the G'(m, l) $\cup X_{11} \cup X_{12}$. Let $l_{131} \cup l_{132} \cup l_{133} \cup X_{11} \cup X_{12} \cup l_{31} \cup l_{41}$ be a path which connects points 11, 12, 13, 14 and meridian curves m_1, m_2, m_3, m_4. Then the fundamental group is as follows;

$$\pi_1(S^3)=\left\langle \begin{array}{l|l} l_{11},\ l_{12},\ l_{131},\ l_{132} & l_{14}l_{22}^{-1}=1,\ l_{22}l_{12}^{-1}=1,\ l_{14}l_{15}^{-1}l_{23}l_{12}^{-1}=1 \\ l_{133},\ l_{14},\ l_{15} & l_{11}l_{133}^{-1}l_{42}l_{42}^{-1}l_{41}^{-1}l_{41}l_{132}^{-1}l_{32}l_{32}^{-1} \\ l_{21},\ l_{22},\ l_{23} & \to l_{31}^{-1}l_{31}l_{131}^{-1}l_{23}l_{15}^{-1}=1 \\ l_{31},\ l_{32},\ l_{41},\ l_{42} & l_{11}l_{21}^{-1}=1,\ l_{21}l_{133}^{-1}l_{132}^{-1}l_{131}^{-1}=1 \\ & l_{11}l_{12}l_{131}l_{132}l_{133}l_{14}l_{15}=1 \\ & l_{21}l_{22}l_{23}=1,\ l_{31}l_{32}=1,\ l_{41}l_{42}=1 \\ & l_{131}=1,\ l_{132}=1,\ l_{133}=1,\ l_{31}=1,\ l_{41}=1 \end{array} \right\rangle$$

Corollary 3. *If* $\bigcup_{i=1}^{g} \{\partial X_{i1}, \cdots, \partial X_{it_i}\} \subset \bigcup_{j=1}^{n} (m_{j1} \cup \cdots \cup m_{jl_j})$ *then any* 1*-cells* $\{l_{ij}\}$ *are not decomposed into* 1*-cells. Hence the presentations of* (*a*) *and* (*b*) *become more simplicity as follows;*

(*a*) $\pi_1(M^3)=\langle l_{ij},\ X_{kl} \mid L_1=1,\ \cdots,\ L_{n-1}=1,\ r_1=1,\ \cdots,\ r_p=1,\ l_{i1}\cdots l_{ik_i}=1\rangle$

$$(i=1,\cdots,n,\ k=1,\cdots,g)$$

(*b*)
$$\pi_1(M^3)=\left\langle l_{ij},\ X_{kl} \left| \begin{array}{l} L_1=1,\ \cdots,\ L_h=1,\ r_1=1,\ \cdots,\ r_p=1,\ l_{i1}\cdots l_{ik_i}=1 \\ X_{k1}=1,\ \cdots,\ X_{kt_k}=1 \end{array}\right.\right\rangle$$

$$(i=1,\cdots,n,\ 0 \leqq h < n-1,\ k=1,\cdots,g)$$

$$=\langle l_{ij} \mid L_1=1,\ \cdots,\ L_h=1,\ r_1=1,\ \cdots,\ r_p=1,\ l_{i1}\cdots l_{ik_i}=1 \rangle$$

$$(i=1,\cdots,n,\ 0 \leqq h < n-1)$$

Definition 13. Each presentation (A), (B1) or (B2) is called a **presentation of a fundamental group by G(m, l)**, respectively. Similarly a **presentation of a fundamental group by G(l, m)** is defined where G(l, m) is the pair of G(m, l). We denote this presentation as follows;

G(l, m) is connected.

$$\pi_1(M^3)=\langle m_{ji} \mid M_1=1,\ \cdots,\ M_{n-1}=1,\ r_1=1,\ \cdots,\ r_p=1,\ m_{j1}\cdots m_{jl_j}=1\rangle\cdots(\mathrm{A}')$$
$$(j=1,\cdots,n)$$

G(l, m) is disconnected.

(*a*) (U; m, l) is connected.

$$\pi_1(M^3)=$$
$$\left\langle \begin{array}{l|l} m_{ji} & M_1=1,\ \cdots,\ M_{n-1}=1,\ r_1=1,\ \cdots,\ r_p=1,\ m_{j1}\cdots m_{jl_j}=1 \\ m_{xyr} & (M_x=m_{xy}=m_{xy1}\cdots m_{xyr_x}\to)\ m_{xy1}=1,\ \cdots,\ m_{xyr_x}=1 \\ X_{kl} & (m_{xy}=m_{xy1}\cdots m_{xyr_x}(m_{xy}\neq M_k)\to)\ m_{xy1}'=1,\ \cdots,\ m_{xyr_x}'=1 \end{array} \right\rangle\cdots(\mathrm{B}1')$$
$$(j=1,\cdots,n-\beta,\ x=1,\cdots,\beta,\ k=1,\cdots,g)$$

(*b*) (U; m, l) is disconnected.

$$\pi_1(M^3)=$$
$$\left\langle \begin{array}{l|l} m_{ji} & M_1=1,\ \cdots,\ M_h=1,\ r_1=1,\ \cdots,\ r_p=1,\ m_{j1}\cdots m_{jl_j}=1 \\ m_{xyr} & (M_x=m_{xy}=m_{xy1}\cdots m_{xyr_x}\to)\ m_{xy1}=1,\ \cdots,\ m_{xyr_x}=1 \\ X_{kl} & (m_{xy}=m_{xy1}\cdots m_{xyr_x}(m_{xy}\neq M_k)\to)\ m_{xy1}'=1,\ \cdots,\ m_{xyr_x}'=1 \\ & X_{k1}=1,\ \cdots,\ X_{kt_k}=1 \end{array} \right\rangle\cdots(\mathrm{B}2')$$
$$(j=1,\cdots,n-\beta,\ x=1,\cdots,\beta,\ 0\leqq h<n-1,\ k=1,\cdots,g)$$

$$=\left\langle \begin{array}{l|l} m_{ji} & M_1=1,\ \cdots,\ M_h=1,\ r_1=1,\ \cdots,\ r_p=1,\ m_{j1}\cdots m_{jl_j}=1 \\ m_{xyr} & (M_x=m_{xy}=m_{xy1}\cdots m_{xyr_x}\to)\ m_{xy1}=1,\ \cdots,\ m_{xyr_x}=1 \\ & (m_{xy}=m_{xy1}\cdots m_{xyr_x}(m_{xy}\neq M_k)\to)\ m_{xy1}'=1,\ \cdots,\ m_{xyr_x}'=1 \end{array} \right\rangle$$
$$(j=1,\cdots,n-\beta,\ x=1,\cdots,\beta,\ 0\leqq h<n-1)$$

A presentation of a fundamental group which is known from a meridian, longitude system of a genus n Heegaard diagram (U; m, l) of (U, V, F) is as follows;

Let each m_i, l_i be oriented. While we take a turn round the circle l_j according to the orientation of l_j, we read continuously the label m_i of the circle as m_i^{+1} (resp. m_i^{-1}) if l_j cross m_i from the upper part (resp. the underpart) of m_i to downward (resp. upward) of m_i. See figure 6. Note that we may start reading from an any cross point of

$\{l_j \cap (m_1 \cup \cdots \cup m_n)\}$.

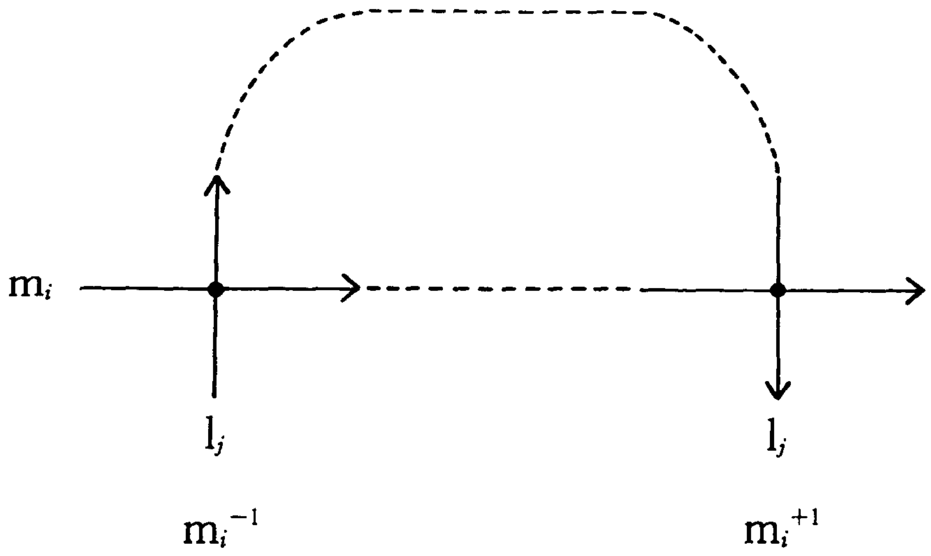


Figure 6

Then we can obtain a cyclic word. Let the word be $\hat{l}_j$. Then $m_1, \cdots, m_n$ become generators and $\hat{l}_1=1, \cdots, \hat{l}_n=1$ become relations of a fundamental group of M^3. We denote this presentation as follows;

$\pi_1(M^3)=< m_1, \cdots, m_n \mid \hat{l}_1=1, \cdots, \hat{l}_n=1 > \cdots$ (m)

A dual presentation from (V; l, m) of (U, V, F) is also defined in an analogous manner and it is denoted as follows;

$\pi_1(M^3)=< l_1, \cdots, l_n \mid \hat{m}_1=1, \cdots, \hat{m}_n=1 > \cdots$ (l).

Definition 14. Each presentation (m), (l) of a fundamental group is called a **presentation of a fundamental group by a meridian, longitude system** of (U; m, l), (V; l, m), respectively.

Each presentation (m), (l) is directly obtained from (U; m, l), (V; l, m), respectively even if each Heegaard diagram is disconnected. But to obtain the presentations (B1), (B1'), (B2) or (B2') we must construct 2-cells in G(m, l) or G(l, m).

§4. Proofs of theorem A and B.

To prove theorem A and B, we will apply a method of the presentation of the fundamental group from a polygram which is given in [1] to $(G(m, l) \cup G(l, m), f)$ or
$((G'(m, l) \cup (\bigcup_{i=1}^{g} (X_{i1} \cup \cdots \cup X_{it_i}))) \cup (G'(l, m) \cup (\bigcup_{i=1}^{g} (X_{i1} \cup \cdots \cup X_{it_i}))), f)$, respectively.

Lemma. *Let (U; m, l) be a genus $n(\geqq 1)$ Heegaard diagram of (U, V, F) of M^3. Let G(m, l) be a connected subdivision of (U; m, l) and G(m, l) be* *the same presentation as in definition* 5. *Let $WG(l)$ be W-graph of the $G(m, l)$. Then a fundamental group of M^3 is obtained from $WG(l)$ or $G(m, l)$.*

Proof. There is G(m, l) (resp. G(l, m)) in the face S_U^2 of B_U^3 (resp. S_V^2 of B_V^3). M^3 is obtained by identifying the same labels of 0-, 1- or 2-cells in $S_U^2 \cup S_V^2$. Hence $f \mid (S_U^2 \cup S_V^2)$ becomes surface complex F^2. F^2 consists of 2-cells $(\sigma_1 \cup \cdots \cup \sigma_p) \cup (D_1 \cup \cdots \cup D_n) \cup (D_1' \cup \cdots \cup D_n')$. Let O be a center point of the meridian disk D_1 of U and O be a base point of a fundamental group of M^3. Then by applying the van Kampen theorem to $M^3 = B_U^3 \cup B_V^3$, we can obtain $\pi_1(M^3, O) \approx \pi_1(F^2, O)$. Next we deform F^2 such that each disk D_i in F^2 contract to the center point P_i of D_i. Where P_1 is especially the base point O. Let such deformed surface of F^2 be $F^{2\prime}$. Then we can obtain $\pi_1(F^2, O) \approx \pi_1(F^{2\prime}, O)$. As this deformation, each 2-cell σ_i, D_i' in F^2 is deformed into 2-cell in $F^{2\prime}$. Let each deformed 2-cell of σ_i be σ_i' $(i=1,\cdots,p)$. And let each deformed 2-disk of D_i' be D_i'' $(i=1,\cdots,n)$. Then $F^{2\prime}$ consists of 2-cells $(\sigma_1' \cup \cdots \cup \sigma_p') \cup (D_1'' \cup \cdots \cup D_n'')$. And $(\partial\sigma_1' \cup \cdots \cup \partial\sigma_p') \cup (\partial D_1'' \cup \cdots \cup \partial D_n'')$ becomes a connected W-graph $WG(l) = \bigcup_{i=1}^{n} (l_{i1} \cup \cdots \cup l_{ik_i})$. Vertices of WG(l) are points $\{P_1, \cdots, P_n\}$. A presentation of the fundamental group $\pi_1(F^{2\prime}, O)$ can be determined in an analogous manner in [1], §62. That is, the 1-cells $\{l_{ij}\}$ in $F^{2\prime}$ or, rather, the closed paths in $F^{2\prime}$ corresponding to these 1-cells $\{l_{ij}\}$ become generators of $\pi_1(F^{2\prime}, O)$. The group relations are obtained by running around each $\partial\sigma_i'$ and $\partial D_i''$. We can read each $\partial\sigma_i'$ from WG(l) but we can not read each $\partial D_i''$ from WG(l). However, since $\partial D_i''$ is $l_{i1} \cup \cdots \cup l_{ik_i}$ we can obtain the relation from $\partial D_i''$. Hence $\pi_1(M^3, O)$ is obtained from WG(l). Furthermore $\pi_1(M^3, O)$ is obtained from G(m, l) by reading 1-cells $\{l_{ij}\}$ as omitting 1-cells $\{m_{ij}\}$. Q. E. D.

Proof of theorem A. Proof also give a manner of the more detailed presentation of $\pi_1(F^{2'}, O)$ in lemma.

First we will construct the generators of $\pi_1(F^{2'}, O)$. WG(l) is connected and $L_1 \cup \cdots \cup L_{n-1}$ connects the points $\{P_1, \cdots, P_n\}$. We construct the closed paths in $F^{2'}$ corresponding to these 1-cells $\{l_{ij}\}$ from $\{L_1, \cdots, L_{n-1}\}$ as follows; ∂l_{ij} are two points. Since 1-cell l_{ij} is oriented, let P_{ij} be the initial point of l_{ij} and P_{ij+1} be the terminal point of l_{ij}. Note that P_{ij} or P_{ij+1} sometimes becomes the base point O. Let a path which start from the base point O and reach to P_{ij} be $o(\tilde{L})_{P_{ij}}$ where $\tilde{L}$ is $\Pi L^{\varepsilon_i}_i$, $\varepsilon_i = \pm$ records the ordered array of the orientation of 1-cell L_i in $\tilde{L}$ if we follow the path $\tilde{L}$ from O to P_{ij}. Then the closed path in $F^{2'}$ corresponding to 1-cell l_{ij} is described as $o(\tilde{L})_{P_{ij}} \cdot (l_{ij}) \cdot (o(\tilde{L})_{P_{ij+1}})^{-1}$. Where $\cdot$ indicates a composition of a path and $(o(\tilde{L})_{P_{ij+1}})^{-1}$ indicates the inverse path of $o(\tilde{L})_{P_{ij+1}}$. Let l_{ij} be a homotopy class of $o(\tilde{L})_{P_{ij}} \cdot (l_{ij}) \cdot (o(\tilde{L})_{P_{ij+1}})^{-1}$ that is $l_{ij} = [o(\tilde{L})_{P_{ij}} \cdot (l_{ij}) \cdot (o(\tilde{L})_{P_{ij+1}})^{-1}]$. Then we regard $\{l_{ij}\}$ as the generators of $\pi_1(F^{2'}, O)$.

Next we will give the group relations of $\pi_1(F^{2'}, O)$.

(1) While we take a turn round the circle $\partial\sigma_i'$ in $F^{2'}$, we read continuously the labels of 1-cells in $\partial\sigma_i'$. Then we can obtain a word $A_{i1} A_{i2} \cdots A_{ik} A_{ik+1} \cdots A_{im} (m \geqq 2)$, where A_{ik} is obtained from the label l_{ik} of 1-cell in $\partial\sigma_i'$ as l_{ik}(resp. l_{ik}^{-1}) if the orientation of 1-cell is same (resp. opposite) as the orientation for running around the $\partial\sigma_i'$. Let $A_{ik} \cap A_{ik+1}$ be Q_{ik+1} and $A_{im} \cap A_{i1}$ be Q_{i1}. Then a loop which take a turn round such as $O \to Q_{i1} \to A_{i1} A_{i2} \cdots A_{ik} A_{ik+1} \cdots A_{im} \to Q_{i1} \to O$ is represented as $o(\tilde{L})_{Q_{i1}} \cdot (A_{i1} A_{i2} \cdots A_{ik} A_{ik+1} \cdots A_{im}) \cdot (o(\tilde{L})_{Q_{i1}})^{-1}$. Then a homotopy class of the loop becomes $[o(\tilde{L})_{Q_{i1}} \cdot (A_{i1} A_{i2} \cdots A_{ik} A_{ik+1} \cdots A_{im}) \cdot (o(\tilde{L})_{Q_{i1}})^{-1}]$. Then this homotopy class is deformed as follows;

$$[o(\tilde{L})_{Q_{i1}}\cdot(A_{i1}A_{i2}\cdots A_{ik}A_{ik+1}\cdots A_{im})\cdot(o(\tilde{L})_{Q_{i1}})^{-1}]$$

$$=[o(\tilde{L})_{Q_{i1}}\cdot(A_{i1})\cdot(o(\tilde{L})_{Q_{i2}})^{-1}\ o(\tilde{L})_{Q_{i2}}\cdot(A_{i2})\cdot(o(\tilde{L})_{Q_{i3}})^{-1}\ o(\tilde{L})_{Q_{i3}}\cdot\ \cdots$$
$$\cdot(o(\tilde{L})_{Q_{ik}})^{-1}\ o(\tilde{L})_{Q_{ik}}\cdot(A_{ik})\cdot(o(\tilde{L})_{Q_{ik+1}})^{-1}\ o(\tilde{L})_{Q_{ik+1}}\cdot(A_{ik+1})\cdot\ \cdots$$
$$\cdot(o(\tilde{L})_{Q_{im-1}})^{-1}\ o(\tilde{L})_{Q_{im-1}}\cdot(A_{im})\cdot(o(\tilde{L})_{Q_{i1}})^{-1}]$$

$$=[o(\tilde{L})_{Q_{i1}}\cdot(A_{i1})\cdot(o(\tilde{L})_{Q_{i2}})^{-1}]\ [o(\tilde{L})_{Q_{i2}}\cdot(A_{i2})\cdot(o(\tilde{L})_{Q_{i3}})^{-1}]\ \cdots$$
$$[o(\tilde{L})_{Q_{ik}}\cdot(A_{ik})\cdot(o(\tilde{L})_{Q_{ik+1}})^{-1}]\ \cdots\ [o(\tilde{L})_{Q_{im-1}}\cdot(A_{im})\cdot(o(\tilde{L})_{Q_{i1}})^{-1}]$$

$$=A_{i1}A_{i2}\cdots A_{ik}\cdots A_{im}$$

Hence a relator is obtained from $\partial\sigma_i'$ as a word $A_{i1}A_{i2}\cdots A_{ik}\cdots A_{im}$.

(2) Similarly a relator is obtained from $\partial D_i''$ as a word $l_{i1}l_{i2}\cdots l_{ik_i}$.

(3) Especially if $L_k=l_{ij}$ then L_k becomes the unit element 1 of $\pi_1(F^{2\prime}, O)$. Because the closed path in $F^{2\prime}$ corresponding to L_k is $o(\tilde{L})_{P_{ij}}\cdot(o(\tilde{L})_{P_{ij}})^{-1}$. $o(\tilde{L})_{P_{ij}}\cdot(o(\tilde{L})_{P_{ij}})^{-1}$ becomes homotopic 1 in $F^{2\prime}$. Hence $L_k=1\,(k=1,\cdots,n-1)$ become the relations of $\pi_1(F^{2\prime}, O)$. Similarly if $L_k=l_{ij}^{-1}$ then $L_k^{-1}=1$ $(k=1,\cdots,n-1)$ become the relations of $\pi_1(F^{2\prime}, O)$. Q. E. D.

If Heegaard genus n equal to 1 then we do not need to choose such L_i. Hence we can obtain corollary 1.

Proof of theorem B. Let G(l, m) be a pair of the G(m, l). Then $((G'(m,\ l)\cup(\bigcup_{i=1}^{g}(X_{i1}\cup\cdots\cup X_{it_i})))\cup(G'(l,\ m)\cup(\bigcup_{i=1}^{g}(X_{i1}\cup\cdots\cup X_{it_i}))),\ f)$ satisfies the conditions of a polygram without connectedness. Let WG'(l) be the W-graph of G'(m, l). Let F^2 be a surface complex which is obtained by f as in the proof in lemma. Then we can obtain $\pi_1(M^3, O) \approx \pi_1(F^2, O)$. Furthermore if we construct a surface $F^{2'}$ from F^2 then we can obtain $\pi_1(F^2, O) \approx \pi_1(F^{2'}, O)$. WG(l) is disconnected but $WG'(l) \cup (\bigcup_{i=1}^{g} (X_{i1} \cup \cdots \cup X_{it_i}))$ becomes connected. Hence $\pi_1(F^{2'}, O)$ is obtained by reading 1-cells in $WG'(l) \cup (\bigcup_{i=1}^{g} (X_{i1} \cup \cdots \cup X_{it_i}))$.

(a) Proof is carried out as the proof of theorem A.

(b) Proof is also carried out as the proof of theorem A and by adding $\bigcup_{i=1}^{g} \{X_{i1}, \cdots, X_{it_i}\}$ to WG(l), we can construct a path which connect the points $\{P_1, \cdots, P_n\}$. Hence $X_{i1}=1, \cdots, X_{it_i}=1\ (i=1,\cdots,g)$ are relations of $\pi_1(F^{2'}, O)$. Q. E. D.

§5. Examples.

We will give some examples which are stated in the introduction.

Example 8. Figure 7 gives genus 2 Heegaard diagrams (U; m, l) and (V; l, m) of (U, V, F) of the Poincaré space and its subdivisions G(m, l) and G(l, m), respectively. This Poincaré space is given by H. Poincaré in 1904.

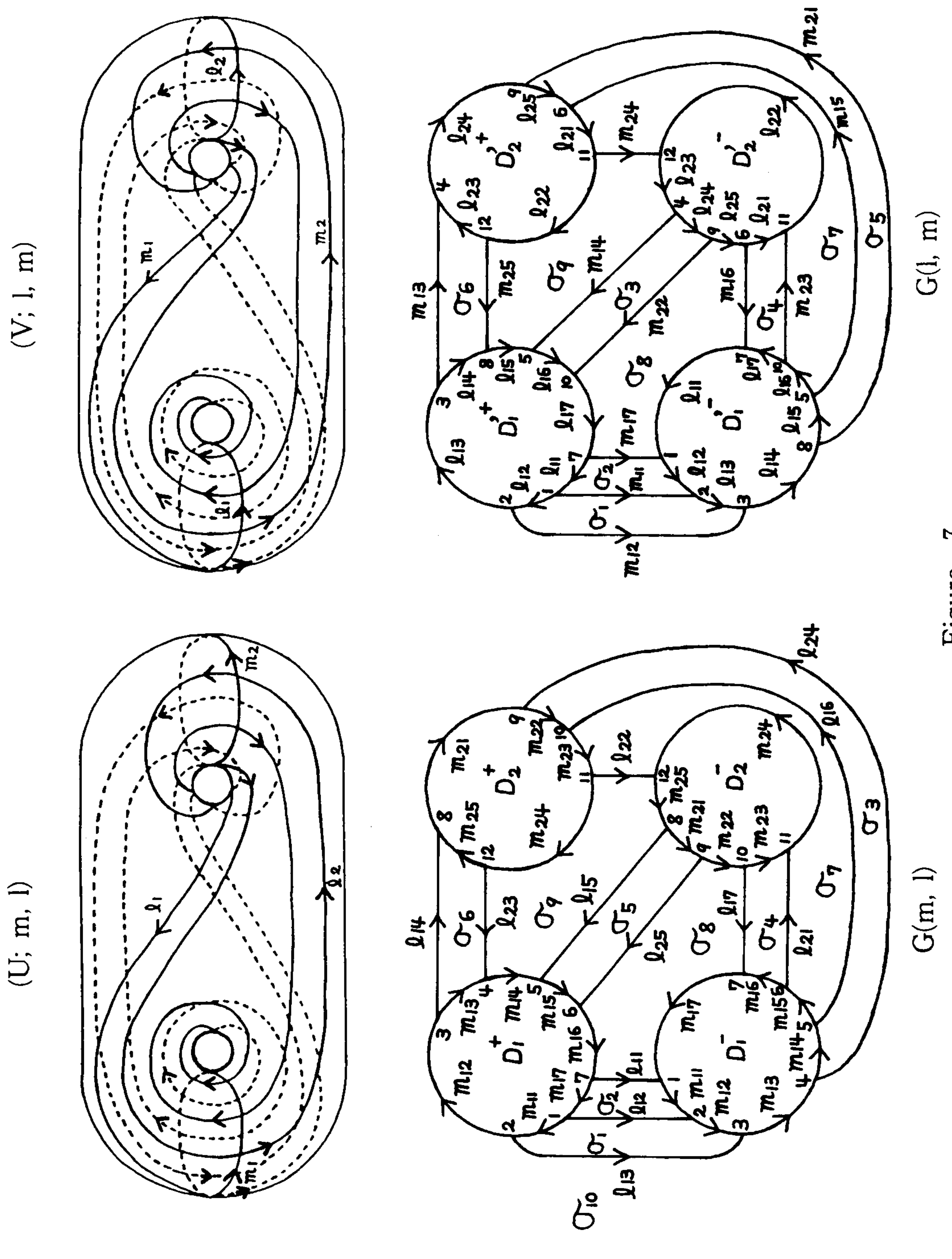


Figure 7

A fundamental group obtained from (U; m, l) is

$$\pi_1 = < m_1, m_2 \mid m_2^{-1} m_1^4 m_2^{-1} m_1^{-1} = 1,\ m_2^2 m_1^{-1} m_2^{-1} m_1^{-1} = 1 >.$$

And a fundamental group obtained from G(m, l) is as follows; now let l_{14} be L_1 then we get

$$\pi_1 = \left\langle \begin{array}{l|l} l_{11}, l_{12}, l_{13}, l_{14}, l_{15}, l_{16}, l_{17} & L_1 = l_{14} = 1,\ l_{12} = l_{13},\ l_{11} = l_{12},\ l_{24} = l_{16} \\ l_{21}, l_{22}, l_{23}, l_{24}, l_{25} & l_{21} = l_{17}^{-1},\ l_{25} = l_{15},\ l_{23} = l_{14}^{-1} \\ & l_{21} l_{22}^{-1} l_{16}^{-1} = 1,\ l_{25} l_{11} l_{17}^{-1} = 1 \\ & l_{22} l_{15} l_{23}^{-1} = 1,\ l_{13} l_{24} l_{14}^{-1} = 1 \\ & l_{11} l_{12} l_{13} l_{14} l_{15} l_{16} l_{17} = 1 \\ & l_{21} l_{22} l_{23} l_{24} l_{25} = 1 \end{array} \right\rangle .$$

We carry out calculations of substitutions in the relations repeatedly, as follows;

$$\approx \left\langle \begin{array}{l|l} l_{11}, l_{12}, l_{13}, l_{15}, l_{16}, l_{17} & l_{11} = l_{12} = l_{13},\ l_{24} = l_{16},\ l_{21} = l_{17}^{-1},\ l_{25} = l_{15},\ l_{23} = 1 \\ l_{21}, l_{22}, l_{23}, l_{24}, l_{25} & l_{21} l_{22}^{-1} l_{16}^{-1} = 1,\ l_{25} l_{11} l_{17}^{-1} = 1,\ l_{22} l_{15} l_{23}^{-1} = 1 \\ & l_{13} l_{24} = 1,\ l_{11} l_{12} l_{13} l_{15} l_{16} l_{17} = 1 \\ & l_{21} l_{22} l_{23} l_{24} l_{25} = 1 \end{array} \right\rangle$$

$$\approx \left\langle \begin{array}{l|l} l_{11}, l_{15}, l_{16}, l_{17} & l_{17}^{-1} l_{22}^{-1} l_{16}^{-1} = 1,\ l_{15} l_{11} l_{17}^{-1} = 1,\ l_{22} l_{15} = 1,\ l_{11} l_{16} = 1 \\ l_{22} & l_{11} l_{11} l_{11} l_{15} l_{16} l_{17} = 1,\ l_{17}^{-1} l_{22} l_{16} l_{15} = 1 \end{array} \right\rangle$$

$$\approx \left\langle \begin{array}{l|l} l_{11}, l_{15}, l_{16}, l_{17} & l_{17}^{-1} l_{15} l_{11} = 1,\ l_{15} l_{11} l_{17}^{-1} = 1 \\ l_{22} & l_{11} l_{11} l_{11} l_{15} l_{11}^{-1} l_{17} = 1,\ l_{17}^{-1} l_{15}^{-1} l_{11}^{-1} l_{15} = 1 \end{array} \right\rangle$$

$$\approx < l_{11},\ l_{15}\ \mid l_{11}l_{11}l_{11}l_{15}l_{11}^{-1}l_{15}l_{11}=1,\ l_{11}^{-1}l_{15}^{-1}l_{15}^{-1}l_{11}^{-1}l_{15}=1 >$$

$$\approx < l_{11},\ l_{15}\ \mid l_{11}^{4}l_{15}l_{11}^{-1}l_{15}=1,\ l_{11}^{-1}l_{15}^{-2}l_{11}^{-1}l_{15}=1 >$$

The last presentation is the same as the presentation from (U; m, l).

Example 9. Figure 8 gives genus 2 Heegaard diagrams (U; m, l) and (V; l, m) of (U, V, F) of the dodecahedron space and its subdivisions G(m, l) and G(l, m), respectively.

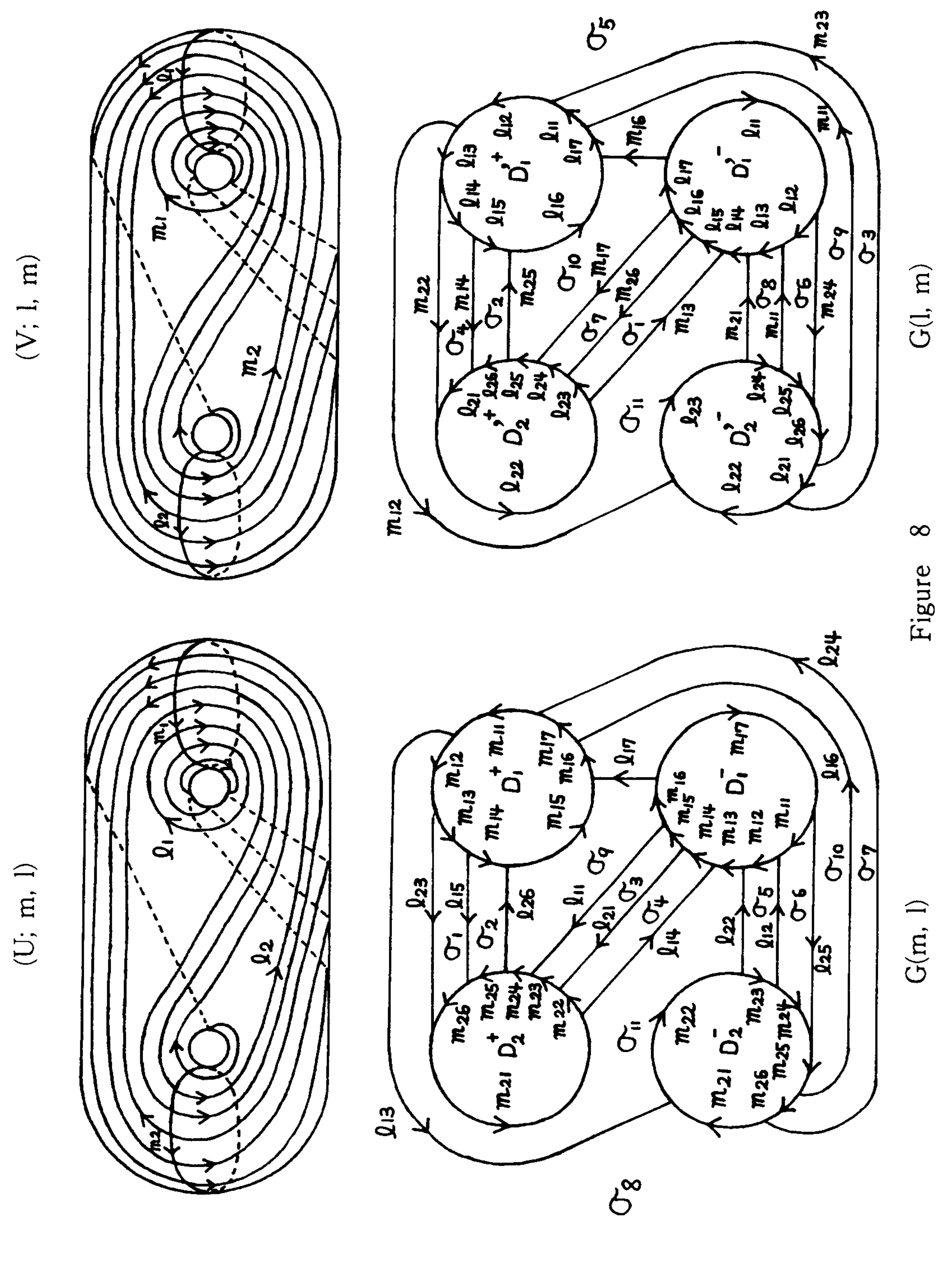


Figure 8

A fundamental group obtained from (U; m, l) is

$$\pi_1 = < m_1, m_2 \mid m_1 m_2 m_1^{-1} m_2^{-1} m_1^{-1} m_2 m_1 = 1,\ m_1 m_2 m_1^{-1} m_2 m_1 m_2^{-1} = 1 >.$$

And a fundamental group obtained from G(m, l) is as follows; now let l_{11} be L_1 then we get

$$\pi_1 = \left\langle \begin{array}{l|l} l_{11}, l_{12}, l_{13}, l_{14}, l_{15}, l_{16}, l_{17} & L_1 = l_{11} = 1,\ l_{15} = l_{23},\ l_{15} = l_{26}^{-1},\ l_{11} = l_{21} \\ l_{21}, l_{22}, l_{23}, l_{24}, l_{25}, l_{26} & l_{14} = l_{21}^{-1},\ l_{12} = l_{22},\ l_{12} = l_{25}^{-1},\ l_{16} = l_{24} \\ & l_{13} = l_{24}^{-1},\ l_{11} l_{26} l_{17}^{-1} = 1 \\ & l_{25} l_{16} l_{17}^{-1} = 1,\ l_{23} l_{14} l_{22}^{-1} l_{13}^{-1} = 1 \\ & l_{11} l_{12} l_{13} l_{14} l_{15} l_{16} l_{17} = 1 \\ & l_{21} l_{22} l_{23} l_{24} l_{25} l_{26} = 1 \end{array} \right\rangle.$$

We carry out calculations of substitutions in the relations repeatedly, as follows;

$$\approx \left\langle \begin{array}{l|l} l_{12}, l_{13}, l_{15}, l_{16}, l_{17} & l_{15} = l_{23},\ l_{15} = l_{26}^{-1},\ l_{12} = l_{22},\ l_{12} = l_{25}^{-1},\ l_{16} = l_{24} \\ l_{22}, l_{23}, l_{24}, l_{25}, l_{26} & l_{13} = l_{24}^{-1},\ l_{26} l_{17}^{-1} = 1,\ l_{25} l_{16} l_{17}^{-1} = 1 \\ & l_{23} l_{22}^{-1} l_{13}^{-1} = 1,\ l_{12} l_{13} l_{15} l_{16} l_{17} = 1 \\ & l_{22} l_{23} l_{24} l_{25} l_{26} = 1 \end{array} \right\rangle$$

$$\approx \left\langle \begin{array}{l|l} l_{12}, l_{13}, l_{15}, l_{16} & l_{12} = l_{22},\ l_{12} = l_{25}^{-1},\ l_{16} = l_{24},\ l_{13} = l_{24}^{-1},\ l_{25} l_{16} l_{15} = 1 \\ l_{22}, l_{24}, l_{25} & l_{15} l_{22}^{-1} l_{13}^{-1} = 1,\ l_{12} l_{13} l_{15} l_{16} l_{15}^{-1} = 1 \\ & l_{22} l_{23} l_{24} l_{25} l_{15}^{-1} = 1 \end{array} \right\rangle$$

$$\approx \left\langle \begin{array}{l|l} l_{12}, l_{13}, l_{15}, l_{16} & l_{16} = l_{24},\ l_{13} = l_{24}^{-1},\ l_{12}^{-1} l_{16} l_{15} = 1,\ l_{15} l_{12}^{-1} l_{13}^{-1} = 1 \\ l_{24} & l_{12} l_{13} l_{15} l_{16} l_{15}^{-1} = 1,\ l_{12} l_{23} l_{24} l_{12}^{-1} l_{15}^{-1} = 1 \end{array} \right\rangle$$

$$\approx \left\langle l_{12},\ l_{15},\ l_{24} \;\middle|\; \begin{array}{l} l_{12}^{-1}l_{24}l_{15}=1,\ l_{15}l_{12}^{-1}l_{24}=1,\ l_{12}l_{24}^{-1}l_{15}l_{24}l_{15}^{-1}=1 \\ l_{12}l_{15}l_{24}l_{12}^{-1}l_{15}^{-1}=1 \end{array} \right\rangle$$

$$\approx \left\langle l_{12},\ l_{15},\ l_{24} \;\middle|\; \begin{array}{l} l_{12}^{-1}l_{24}l_{15}=1,\ l_{12}l_{24}^{-1}l_{15}l_{24}l_{15}^{-1}=1 \\ l_{12}l_{15}l_{24}l_{12}^{-1}l_{15}^{-1}=1 \end{array} \right\rangle$$

$$\approx < l_{15},\ l_{24} \mid l_{24}l_{15}l_{24}^{-1}l_{15}l_{24}l_{15}^{-1}=1,\ l_{24}l_{15}l_{15}l_{24}l_{15}^{-1}l_{24}^{-1}l_{15}^{-1}=1 >$$

The last presentation is the same as the presentation from (U; m, l).

Next if we change from $L_1=l_{11}=1$ into $L_1=l_{23}=1$ in the first presentation then its presentation becomes

$$\pi_1=\left\langle \begin{array}{l} l_{11},\ l_{12},\ l_{13},\ l_{14},\ l_{15},\ l_{16},\ l_{17} \\ l_{21},\ l_{22},\ l_{23},\ l_{24},\ l_{25},\ l_{26} \end{array} \;\middle|\; \begin{array}{l} L_1=l_{23}=1,\ l_{15}=l_{23},\ l_{15}=l_{26}^{-1},\ l_{11}=l_{21} \\ l_{14}=l_{21}^{-1},\ l_{12}=l_{22},\ l_{12}=l_{25}^{-1},\ l_{16}=l_{24} \\ l_{13}=l_{24}^{-1},\ l_{11}l_{26}l_{17}^{-1}=1 \\ l_{25}l_{16}l_{17}^{-1}=1,\ l_{23}l_{14}l_{22}^{-1}l_{13}^{-1}=1 \\ l_{11}l_{12}l_{13}l_{14}l_{15}l_{16}l_{17}=1 \\ l_{21}l_{22}l_{23}l_{24}l_{25}l_{26}=1 \end{array} \right\rangle.$$

We carry out calculations of substitutions in the relations repeatedly, as follows;

$$\approx \left\langle \begin{array}{l} l_{11},\ l_{12},\ l_{13},\ l_{14},\ l_{16},\ l_{17} \\ l_{21},\ l_{22},\ l_{24},\ l_{25} \end{array} \;\middle|\; \begin{array}{l} l_{11}=l_{21},\ l_{14}=l_{21}^{-1},\ l_{12}=l_{22},\ l_{12}=l_{25}^{-1},\ l_{16}=l_{24} \\ l_{13}=l_{24}^{-1},\ l_{11}=l_{17},\ l_{25}l_{16}=l_{17},\ l_{14}l_{22}^{-1}=l_{13} \\ l_{11}l_{12}l_{13}l_{14}l_{16}l_{17}=1,\ l_{21}l_{22}l_{24}l_{25}=1 \end{array} \right\rangle$$

$$\approx \left\langle \begin{array}{l} l_{11},\ l_{12},\ l_{13},\ l_{16} \\ l_{22},\ l_{24},\ l_{25} \\ \end{array} \;\middle|\; \begin{array}{l} l_{12}=l_{22},\ l_{12}=l_{25}^{-1},\ l_{16}=l_{24},\ l_{13}=l_{24}^{-1},\ l_{25}l_{16}=l_{11} \\ l_{11}^{-1}=l_{22}^{-1}=l_{13},\ l_{11}l_{12}l_{13}l_{11}^{-1}l_{16}l_{11}=1 \\ l_{11}l_{22}l_{24}l_{25}=1 \end{array} \right\rangle$$

$$\approx \left\langle \begin{array}{l} l_{11},\ l_{12},\ l_{13},\ l_{16} \\ l_{24} \end{array} \;\middle|\; \begin{array}{l} l_{16}=l_{24},\ l_{13}^{-1}=l_{24},\ l_{12}^{-1}l_{16}=l_{11},\ l_{11}^{-1}l_{12}^{-1}=l_{13} \\ l_{11}l_{12}l_{13}l_{11}^{-1}l_{16}l_{11}=1,\ l_{11}l_{12}l_{24}l_{12}^{-1}=1 \end{array} \right\rangle$$

$$\approx \left\langle \begin{array}{l} l_{11},\ l_{12},\ l_{24} \end{array} \;\middle|\; \begin{array}{l} l_{12}^{-1}l_{24}=l_{11},\ l_{11}^{-1}l_{12}^{-1}=l_{24}^{-1},\ l_{11}l_{12}l_{24}^{-1}l_{11}^{-1}l_{24}l_{11}=1 \\ l_{11}l_{12}l_{24}l_{12}^{-1}=1 \end{array} \right\rangle$$

$$\approx < l_{12},\ l_{24} \mid l_{12}^{-1}l_{24}l_{12}l_{24}^{-1}l_{24}^{-1}l_{12}l_{24}l_{12}^{-1}l_{24}=1,\ l_{12}^{-1}l_{24}l_{12}l_{24}l_{12}^{-1}=1 >$$

$$\approx < l_{12},\ l_{24} \mid l_{12}l_{24}^{-1}l_{24}^{-1}l_{24}^{-1}l_{12}l_{24}l_{12}^{-1}l_{24}=1,\ l_{12}^{-1}l_{24}l_{12}=l_{12}l_{24}^{-1} >$$

$$\approx < l_{12},\ l_{24} \mid l_{12}^{-1}l_{24}l_{12}l_{24}^{-1}l_{24}^{-1}l_{24}^{-1}l_{12}l_{24}=1,\ l_{12}^{-1}l_{24}l_{12}=l_{12}l_{24}^{-1} >$$

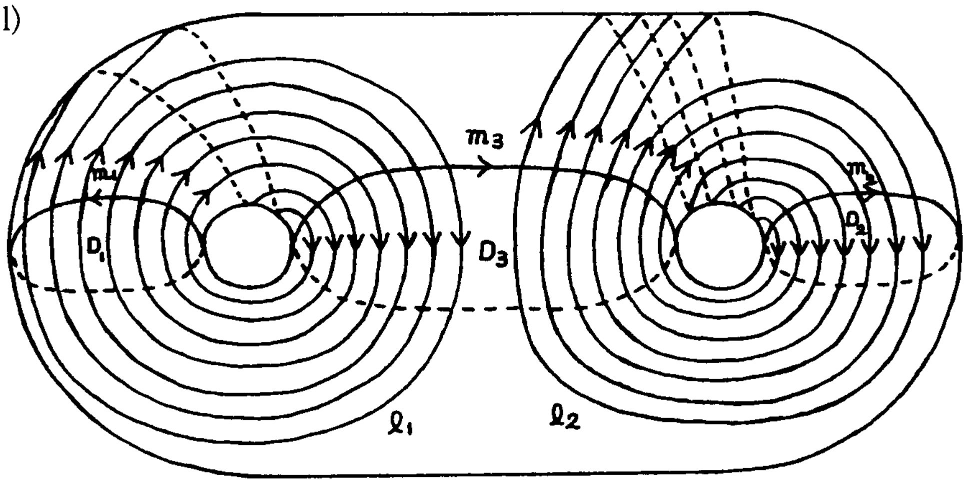
$$\approx < l_{12},\ l_{24} \mid l_{12}l_{24}^{-4}l_{12}l_{24}=1,\ l_{12}^{-1}l_{24}l_{12}l_{24}l_{12}^{-1}=1 >$$

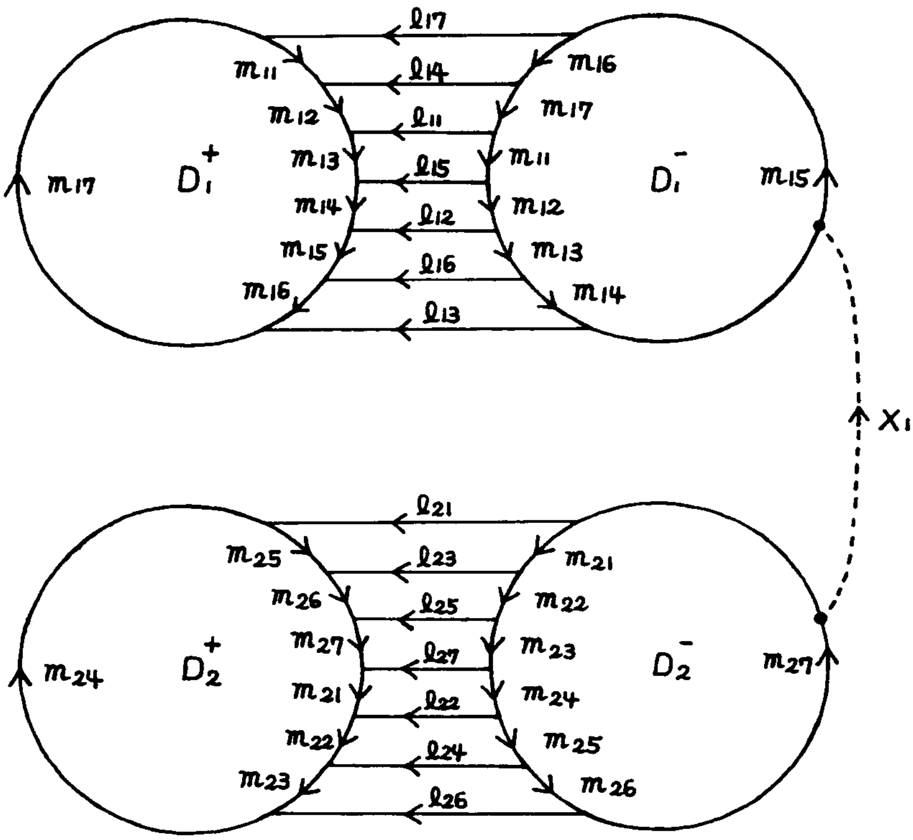
$$\approx < l_{12},\ l_{24} \mid l_{12}l_{24}^{-4}l_{12}l_{24}=1,\ l_{12}^{-2}l_{24}l_{12}l_{24}=1 >$$

The last presentation is the same as the presentation from (U; m, l) of the Poincaré space in example 8.

Example 10. Figure 9 gives a disconnected genus 2 Heegaard diagram (U; m, l) of (U, V, F) of L(7, 2)#L(7, 4) and its subdivision G(m, l) without dotted line X_1. Where # denotes the connected sums of L(7, 2) and L(7, 4).

(U; m, l)





G(m, l)

Figure 9

A fundamental group obtained from (U; m, l) is

$\pi_1(L(7, 2)\# L(7, 4)) = < m_1, m_2 \mid m_1^{-7}=1, m_2^7=1 > \approx Z_7 * Z_7 \approx$

$\pi_1(L(7, 2)) * \pi_1(L(7, 4))$ ($*$ denotes free product group).

And a fundamental group obtained from $G'(m, l) \cup X_1$ is

$\pi_1(L(7, 2)\# L(7, 4)) \approx$

$$\left\langle \begin{array}{l|l} l_{11}, l_{12}, l_{13}, l_{14}, l_{15}, l_{16}, l_{17} & l_{13}l_{16}^{-1}=1,\ l_{16}l_{12}^{-1}=1,\ l_{12}l_{15}^{-1}=1 \\ l_{21}, l_{22}, l_{23}, l_{24}, l_{25}, l_{26}, l_{27} & l_{15}l_{11}^{-1}=1,\ l_{11}l_{14}^{-1}=1,\ l_{14}l_{17}^{-1}=1 \\ & l_{26}l_{24}^{-1}=1,\ l_{24}l_{22}^{-1}=1,\ l_{22}l_{27}^{-1}=1 \\ & l_{27}l_{25}^{-1}=1,\ l_{25}l_{23}^{-1}=1,\ l_{23}l_{21}^{-1}=1 \\ & l_{17}l_{13}^{-1}l_{21}l_{26}^{-1}=1 \\ & l_{11}l_{12}l_{13}l_{14}l_{15}l_{16}l_{17}=1 \\ & l_{21}l_{22}l_{23}l_{24}l_{25}l_{26}l_{27}=1 \end{array} \right\rangle$$

$\approx < l_{11}, l_{21} \mid l_{11}^7=1, l_{21}^7=1 > \approx Z_7 * Z_7.$

Example 11. In (U; m, l) in figure 9, if we cut off U at D_2, D_3 then we can obtain a connected subdivision G(m, l). We can also obtain the same result as example 10 from the G(m, l). We leave these works for readers.

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